Proceedings of the 22th ProMath conference
from August 22-24, 2022 in Thessaloniki

PROBLEM SOLVING& PROBLEM POSING:

PERSPECTIVES AND POTENTIALITIES IN RESEARCH AND PRACTICE

IOANNIS PAPADOPOULOS, NAFSIKA PATSIALA (EDS.)
Aristotle University of Thessaloniki, Faculty of Education

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Problem solving in Mathematics Education

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PREFACE

The School of Education at the Aristotle University of Thessaloniki (Greece) hosted the 22nd meeting of the ProMath (**Pro**blem Solving in **Math**ematics Education) community from the 22nd to the 24th of August 2022.

Problem solving and problem posing play crucial role in the development of mathematical thinking. Their importance is confirmed by the attention paid to them by so many researchers all over the world. However, it is also acknowledged that the research findings are not always transferred to real classroom settings. This is why the theme of the conference was "*Problem solving and problem posing: perspectives and potentialities in research and practice*".

Dr Igor Kontorovich (University of Auckland, New Zealand) launched the scientific program with his plenary talk "*Do educational problem-posing efforts align with practices of expert problem posers? Does it matter?*", aiming to bring forward differences and similarities between practices of professional and educational problem posing.

An interesting spectrum of eleven countries –Croatia, Finland, Germany, Greece, Hungary, Israel, New Zealand, Romania, Serbia, Turkey, and United Kingdom– were represented by thirty attending participants presenting their fourteen submissions covering a variety of different aspects of the conference's theme, with research reports addressing both problem-posing and problem-posing topics.

For the purposes of this book, research reports formed two groups for problem solving focusing to teachers and students respectively, and one group for problem posing. All papers have undergone the peer-review process.

Group 1 — Placing teachers at the center: Comparison of problem-solving teaching practices in different settings (Gebel, I., Kuzle, A., Laine, A., & Sturn, N.); difficulties teachers face during problem solving (Antunović-Piton & Baranović); teachers' problem-solving strategies (Milinković); and the role of problem solving in teaching scenarios (Souralis & Triantafyllou).

Group 2 — Investigating problem solving from the students' point of view: The role of technology in problem solving (Thoma, G., Bahnmueller, J., & Moeller, K.); the role of algorithmic skills in problem solving (Kónya, E., & Kovács, Z.); the issue of control in decision making (Ambrus, A., & Kiss, M.); and how problem solving relevant to art might contribute to mathematical learning (Toth, G.).

Group 3 — Examining problem posing: potential link between the seeking-and-using-structure habit of mind and problem posing (Papadopoulos. I, & Patsiala, N.); the link between problem posing and creativity (Zioga, M., & Desli, D); and the students' thinking about problem posing (Báró, E.).

The two presentations in the conference that do not appear in this book investigated the status of mental argumentation in problem solving (Papadopoulos, I., & Papadopoulou, M.) and to what extent problem solving in lesson study might improve teaching of mathematics (Çelebi-İlhan, E. G., Toker, Z., Alkaşulusoy, Ç., Emre-Akdoğan, E., Balci, E., & Güzeller, G.).

The book concludes with the reports from the three workshops that took place during the last day of the conference where the attending participants were prompted in discussing the following issues: (i) "Problem solving and problem posing: autonomous subjects in classroom or integrated in daily mathematics teaching? What about the issue of time allocation?" (ii) "Pedagogical aspects of problem solving and problem posing in classroom", and (iii) "Skills that are possibly developed through an interplay between problem solving and posing in classroom".

Finally, we would like to thank all the volunteers and reviewers, who assisted in various ways before, during, and after the conference. Especially, we would like to thank our designer, Anna Papadopoulou for her inspiring work during the process of both the organization of the conference (logo of the conference, booklet, poster) and the preparation of the proceedings.

Thessaloniki, 02 March 2023

Ioannis Papadopoulos & Nafsika Patsiala

PLENARY TALK

WOULD EXPERIENCED PROBLEM POSERS ENDORSE THE WAY WE ENGAGE NEWCOMERS IN THE ACTIVITY? DOES IT MATTER?[1]

Igor' Kontorovich

The University of Auckland, New Zealand

This paper is concerned with the relationship between problem posing that is carried out in professional communities and an educational version of the activity. Specifically, I aim to generate a discussion on how the former can inform the latter. In this paper, I argue that the educational potential of professional problem posing lays in the conceptual sphere, where it can become a source of new ideas, perspectives, and interpretations, rather than a blueprint for practice. To illustrate the argument, I juxtapose my previous research on how problems for mathematics competitions come about with problem-posing-requiring tasks that are often used to engage mathematics learners in problem posing.

INTRODUCTION

In their book entitled "Evolutions in physics", Einstein and Infeld (1938) offer a range of insights about scientific inquiry. One of them states:

> The formulation of a problem is often more essential than its solution, which may be merely a matter of mathematical or experimental skills. To raise new questions, new possibilities, to regard old questions from a new angle, requires creative imagination and marks real advance in science (p. 92).

Mathematics educators often draw on statements of this sort by eminent scientists and mathematicians to argue for engaging learners in problem posing (e.g., Halmos, 1980; Hilbert, 1901; Cantor and Klamkin's views in Cai & Mamlok-Naaman, 2020). This rhetorical move begs the question – what do the perspectives of exceptional problem posers, and professional problem posers in general, contribute to the educational discourse on problem posing (*PP* hereafter)? This question may appear legitimate to some readers but awkward to others who believe that mathematics learning should emulate the practices of mathematics specialists, especially mathematicians, as a rule. Not to alienate the latter group, let me rephrase the question – if the experiences of professional problem posers are of merit, then how have they been put into use in educational PP?

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1. This is a slightly modified version of the paper that was accepted to Rott, B., Heuer, K., & Baumanns, L. (Eds.), *Problem posing and solving for mathematically gifted and interested students – best practices, research and enrichment*. Springer-Spektrum.

In I. Papadopoulos & N. Patsiala (Eds.), *Proceedings of the 22nd Conference on Problem Solving in Mathematics Education - ProMath 2022* (pp. 3–17). Faculty of Education, Aristotle University of Thessaloniki3

With this conceptual chapter (e.g., Gilson & Goldberg, 2015), I aim to draw mathematics educators' attention to specialist PP and the potential of research on it to advance educational PP. With "specialist", I refer to the whole range of communities that generate mathematical problems as part of their regular professional activity. Educational PP pertains to research and practice that provides students, teachers, and other cohorts of mathematics learners with opportunities to create mathematical problems.

Bridging between specialist and educational PP is an intellectually challenging endeavor since PP is a complex and multi-faceted activity in which many substantially different communities might be construed as specialists. There are also myriad ways to put research on their PP into educational use. I turn to my previous research on experienced problem posers for mathematics competitions (or *EPPMCs* for short) as an example of a community of PP specialists. I use this example to argue that the educational prospects of research on specialist PP lays in the conceptual sphere. Accordingly, there is no "by default" reason for educational PP to "line up" with the norms and practices established in specialist PP communities.

This paper is structured as follows. The next section offers an initial motivation for why mathematics educators may be interested in specialist PP and lays out some caveats of investigating specialists' practices. This is followed by a characterization of tasks and situations that educators often design to engage learners in PP. The next section overviews a part of my previous work on EPPMCs, focusing on what triggers them to pose new problems. By juxtaposing the insights from the two preceding sections, I delineate some similarities and differences between the two types of PP. I conclude by illustrating how attending to similarities and differences between specialist and educational PP can engender new ideas and directions for the latter.

THE CAVEATS OF RESEARCH INTO EDUCATIONAL AND SPECIALIST PP

PP has grown into a prolific research area that has advanced considerably in the last four decades. Taking stock of this research, Cai et al. (2015) argue that it has answered some of the area's pertinent questions to at least some degree. For instance, numerous studies have shown that learners – mostly school students and pre-service teachers – can pose interesting and important mathematical problems (e.g., Cifarelli & Cai, 2005; Crespo & Sinclair (2008); Koichu, 2020; Koichu & Kontorovich, 2013; Silver et al., 1996). Notwithstanding, the area seems to grapple with going beyond the "proof of learners' capability" towards understanding "what it takes" for them to pose quality problems.[2] Indeed, a commonly reported finding suggests that it is not rare for learners to pose "nonmathematical problems, unsolvable problems, and irrelevant problems" (Cai et al., 2015 p. 9). Accordingly, Cai et al. call further research to explore:

- "Why do students pose nonmathematical, trivial, or otherwise suboptimal problems or statements?" (ibid, p. 9);
- "What strategies and ways of thinking are most productive for posing problems, and under what types of mathematical situations are different strategies effective?" (ibid, p. 11);
- "If curriculum designers intend to integrate problem posing into textbooks and teaching materials, what are the best ways to do so?" (ibid, p. 19).

Being interested in promoting quality PP among learners, it seems only reasonable to explore how PP specialists generate their problems. Indeed, it has been a proven practice in the mathematics education community to source insights from research on specialists (mostly mathematicians) as a means to advance mathematics learners (e.g., see Liljedahl, 2009 for mathematical discovery; Schoenfeld, 1992 for problem solving; Wilkerson-Jerde & Wilensky, 2011 for learning new mathematics; Kontorovich, 2015, 2016 for research knowledge in mathematics education). Analogously, studies on specialist PP may be profitable for PP learning and teaching. Overall, it is somewhat curious that the volume of research on specialist PP does not match the educational interest that this activity has evoked over the years: from Kilpatrick's (1987) ambitious agenda for providing opportunities for school students to create their own problems to the institutionalization of these opportunities in the curricula in some countries (e.g., Australia, China, the US). Notably, unlike in PP, specialist problem solving was explored in numerous studies before it found its way into policy documents (for a recent historical review, see Liljedahl & Cai, 2021).

Studying specialist PP is far from straightforward. Weber et al. (2020) deliberate on the challenges that emerge when educators attempt to get insights into authentic mathematical practices. In particular, the researchers point at *the problem of identifying the mathematical community* (i.e., who are the people whose practices are of interest and who is selected to represent these practices?); *the problem of heterogeneity* (i.e., why should one assume that this community engages in these practices similarly?); what I term as *the problem of methodology* as a combination of Weber et al.'s (2020) problems of *the advanced content* (i.e., how educators can comprehend advanced mathematics with which

2. As it may be expected, there is no consensus regarding the notion of quality problem. Using this notion as an umbrella term is sufficient for the purposes of this chapter.

mathematicians operate?) and *the time scale* (i.e., how to study a process that can unfold for a long time?). Two common resolutions of the problem of methodology are task-based interviews and reflection-based methods. However, in the former, mathematicians are asked to engage in a process that can barely count as authentic, when the latter epitomizes the *accuracy problem*. Indeed, several studies have shown that mathematicians and specialists, in general, can be not accurate in their descriptions of their professional deeds (e.g., Inglis & Alcock, 2012; Van Someren et al., 1994).

PP is characteristic not only to mathematicians but to a wide range of specialist communities, including scientists, engineers, modelers, mathematics teachers, teacher educators, textbook and assessment writers, EPPMCs, et cetera. Thus, the problem of recognizing specialist communities whose PP is relevant to educational goals is paramount. Studying PP in each of these communities can entail a subset of problems from Weber et al.'s (2020) list and give rise to community-specific methodological challenges. Yet, for these problems to emerge, disciplined inquiry into specialists' PP must be conducted. To my knowledge, only a handful of studies have pursued this line of inquiry so far. Accordingly, the resulting body of knowledge especially susceptible to criticism (I return to this point shortly).

PP-REQUIRING TASKS AND SITUATIONS

In their comprehensive review of PP, Cai et al. (2015) write, "In this line of research, researchers typically design a problem situation and ask subjects to pose problems which can be solved using the information given in the situation" (p. 6). The usage of such research designs can be traced back to the first empirical investigations (e.g., Ellerton, 1986), and it features in often-cited studies (e.g., Silver et al., 1996; Silver & Cai, 1996). On its face, there seems to be nothing more natural for PP researchers than explicitly asking their participants to generate problems here, now, and preferably out loud. Thus, it is barely surprising that the notions of "situations" and "tasks" are central to this line of research.

The two terms are often used interchangeably in the PP literature, but I propose distinguishing between the two. Borrowing from Mason and Johnston-Wilder (2006), I use "task" to refer to a formal set of instructions that communicates to learners what they are expected to do. In this sense, more often than not, research has been using tasks that can be termed as *PP-requiring* since they explicitly ask participants to come up with mathematical problems. To paraphrase Silver et al. (1996), the goal of these tasks is "the creation of a new problem from a situation or experience" (p. 294). Throughout the years, a considerable number of studies drew attention to the classification of such tasks (e.g., see free, semi-structured, and structured in Stoyanova & Ellerton, 1996), while other studies explored the impact of task formulations on the consequent processes and eventually on the posed problems (e.g., Leung & Silver, 1997).

Brousseau (1997) uses "situation" to account for key elements of the social context (or "milieu") that are necessary to understand a didactical phenomenon. Accordingly, I construe "situation" as a contextual construct that is attentive to a range of circumstances in which learners engage with a task. These circumstances can pertain to situational components that are hard to miss (e.g., do the learners pose individually or in groups?) as well as to more nuanced aspects. These can refer to what the learners are told about the structure of the activity (e.g., are they expected to solve their problems as part or after posing?), whether the learners were informed about the future usage of their problems (e.g., would they be assessed based on the problems they generate, and how, if yes? What was said about the intended audience of the problems), and notable events that preceded the posing phase (e.g., did the invitation to pose come after proving an important theorem?). While many studies are terse regarding the situation in which the participants were tasked with PP, other works provide "thick" descriptions of these situations (e.g., Brown & Walter, 1993), draw on them to interpret the participants' activity (e.g., Koichu & Kontorovich, 2013; Silver et al., 1996), and offer multi-faceted frameworks to account for them (e.g., Kontorovich et al., 2012).

How common is it for teachers to engage learners in PP through PP-requiring situations? The research literature suggests that such situations are used in most if not all PP studies that unfold in mathematics classrooms and special training programs (for a review, see Osana & Pelczer, 2015). That said, in nearly all these settings, the teachers (e.g., Chen et al., 2015) and teacher educators (Leung, 2013; Tichá & Hošpesová, 2012) are PP researchers themselves. Thus, it is barely surprising that these task-setters adhere to methods that have been accepted in PP research.

Various countries have included PP in their mathematics curricula, which has nudged local textbooks to follow. Research on PP opportunities provided by mathematics textbooks is in its infancy, but existing studies suggest that most of these follow a PP-requiring format. For instance, Jia and Yao (2021) report on a historical analysis of PP tasks in primary school textbooks in China. The findings show that the textbook tasks vary in what information they provide for students to pose their problems, but not in the explicit request to pose problems (e.g., "Please make up a word problem using 5+9=14"). Similar results emerge from Cai and Jiang's (2017) study on elementary mathematics textbooks in the US. These studies do not discuss how teachers use the textbook in their mathematics classrooms, or even whether they do so at all. Yet one might hypothesize that many textbook writers resort to PP-requiring tasks as the primary vehicle to promote PP among their student audience.

In PP-requiring tasks and situations, PP often constitutes an "isolated activity where it becomes an end in itself" (English, 2020, p. 3). Drawing on Brousseau (1997), Koichu (2020) proposes an alternative approach, suggesting that learners can pose problems *a-didactically* – "as an activity necessitated for the posers by the need to find or create problems that would serve another goal" (p. 3). Koichu (2020) demonstrates what a-didactical PP may look like in the case of pre-service teachers. Specifically, their task was to develop a teaching sequence intended to prepare regular high-school students to solve a certain type of mathematics competition problem. Posing intermediate problems had to be part of the sequence. An echoing approach emerges from Hartmann et al. (2021), where PP served school students to model real-world scenarios. Note that in these a-didactical situations, the researchers have still put learners in PP-requiring circumstances.

Given sufficient effort, it is possible to find publications that report on learners engaging in PP autonomously. One such publication has come from the Tall family (Tall et al., 2017). It discusses the mathematical growth of young Simon through mathematical conversations with his grandfather, David, and father, Nic. In these conversations, Simon worked on the questions and problems that the adults offered him and reciprocated with mathematical questions and problems of his own. Cifarelli and Cai (2005) engaged students in open-ended problem-solving situations, where they could vary a range of parameters and investigate the consequences. The researchers interpreted students' actions, such as generating particular cases, as problems that they posed to make sense of the assigned situation. In a similar vein, Contreras (2013) reports on an instructional experience with prospective secondary teachers, who raised questions and conjectures, which were reformulated into mathematical problems with the teacher educator's guidance. Drawing on these publications, I suggest that learners can engage in a-didactical PP without being formally asked to generate problems.

WHERE COMPETITION PROBLEMS COME FROM

Research and its problems

My doctoral research was concerned with how EPPMCs come up with mathematical problems. Mathematics Competitions (MCs) have often been described as treasures of "elegant", "intriguing", and "surprising" problems that reach students after thorough discussions in MC committees (e.g., Koichu & Andžāns, 2009). Thus, understanding the posing practices of EPPMCs appeared especially relevant to the educational discourse on teaching and learning to create quality problems.

At that time, I identified less than ten self-reflective publications by EPPMCs and mathematics educators recognized in their respective communities for their PP. On the one hand, almost all of them referred to "sources of mathematical problems". On the other hand, the authors were silent about what they actually meant with this term, the features that turned something into a source, how the posers come across these sources, et cetera (cf. the accuracy problem in Weber et al., 2020). The reflections usually started with a situation where an initial source had already been identified, and then the authors elaborated on specific methods that they used to come up with problems. Overall, these reflections created an impression that, while a source played a vital role in the PP process, its selection was a matter of the poser's deliberate decision. Indeed, Walter (1978) literally argued that problems can be generated almost from anything. These observations set the stage for an empirical study.

Twenty-six EPPMCs participated in that exploratory study. At the time of data collection, their average posing experience was 25 years, and it ranged from 7 to 37 years. The participants were recruited with the snowball technique, which resulted in a cohort that over-represented some aspects and practices in the MC movement while under-representing others (cf. the problem of identification in Weber et al., 2020). Indeed, the participants resided in Australia, Bulgaria, Israel, Latvia, Lithuania, Russian Federation, Spain, Sweden, and the US. They posed problems for national, regional, and international competitions, most of which originated in the Soviet Union, the Russian Federation, or the neighbouring countries. The Russian language was the mother tongue for nearly three-quarters of the participants. Seventeen of them held doctoral degrees in mathematics and two in mathematics education; six participants had a master's in mathematics and one in mathematics education.

To collect data, I asked each EPPMC to pre-select 5 to 10 of their past MC problems, the posing stories of which they remembered in detail. The next step involved individual semi-structured interviews and questionnaires, where the participants were asked to describe how these problems came about in as much detail as possible. I used this data to construct accounts of incidents and experiences that preceded the participants creating their problems. To enhance the credibility of the accounts, a member check technique was employed to allow the participants to modify the drafts of the accounts and better align them with their recollections of events. These steps were taken in an attempt to address the problem of methodology (cf. Weber et al., 2020). The last step consisted of a thematic analysis, where I examined the accounts, interviews, and questionnaires to search for common threads. The analysis aimed at revealing the commonalities between the EPPMCs' practices, while remaining attentive to the differences between them (cf. the problem of heterogeneity in Weber et al., 2020).

EPPMCs' triggers

Building on Mason (2002), in Kontorovich (2020), I introduced the notion of *triggers* as instances of noticing, where an external impulse draws the attention of an EPPMC and "triggers" a mathematical activity that eventually leads them to construct a problem. The term "trigger" was chosen to emphasise that this noticing can be rapid, sub-conscious, and might appeal to the EPPMC's affect-emotional domain. The fact that the activity "triggered" a problem does not necessarily imply that PP was the EPPMC's goal on encountering on the impulse. Next, I elaborate on this point by overviewing three types of triggers that emerged from the data analysis. Due to space limitations, I focus on the aspects that are of direct relevance to educational PP. Interested readers can find illustrations and additional details in the original publication.

In the first trigger, the EPPMCs extracted mathematical phenomena from the mathematics that drew their attention as part of their participation in the MC movement. At different points in their careers, all my participants had led various extracurricular activities for school and university students (e.g., mathematical circles), contributed to campuses preparing students for specific MCs, and served on organizational MC committees. These roles came with a range of characteristic activities, such as keeping up to date with the MC literature, teaching relevant topics to students, and discussing, solving, and posing problems. These activities are replete with mathematics, which frequently served as an external impulse that set the EPPMCs on the path towards new problems.

The mathematics that served as a triggering impulse for the EPPMCs can by no means be described as arbitrary (which is an antipode to Walter's, 1978 thesis). Each participant situated their MC experience within a few domains of mathematical specialization. These domains mostly overlapped with the content areas traditionally associated with MCs (i.e., number theory, algebra, combinatorics, plane, and solid geometry), although some participants framed their domains of specialization in a narrower manner (e.g., mathematical inequalities). Only two participants self-identified as specializing in one domain, and twenty positioned themselves in two domains. The remaining four EPPMCs attested to their experience as being spread among three domains. The participants reported that these domains of specialization shape the literature they engage with, the problems on which they spend their time, and the impulses they tend to notice and seek.

Two aspects characterize the first trigger. They firstly evoke a rich and emotionally loaded lexicon when EPPMCs describe their encounter with the triggering mathematics (e.g., feelings of being overwhelmed, shocked, surprised, and curious). The analysis of the collected accounts suggests that this mathematics disturbed the EPPMCs' mathematical knowledge base in some way. For instance, through challenging something known or illuminating unfamiliar ideas in their domains of specialization. The second aspect concerns the mathematical phenomena discerned by the EPPMCs from the noticed impulses. The formulations of the phenomena extracted by the EPPMCs drew attention to particular ideas that, as one participant put it, "begged for a mathematical investigation". While not prescribing how these investigations should be carried out, for mathematically experienced people, like the EPPMCs, the formulations set up a general course for investigative actions. In other words, the extracted phenomena put the EPPMCs in a position where they could capitalize on their mathematical experience and react in an investigative manner to the impulses that took them out of their comfort zone.

The second trigger emerged from the accounts of three EPPMCs who described their engagement in common everyday-life tasks. If approached with mathematical methods, these tasks could lead to more advantageous consequences. In their accounts of previously posed problems, the EPPMCs highlighted the value that they ascribed to finding an optimal solution, and their desire to compensate for feelings of disappointment at their initial (often sub-optimal) approach. In this way, these situations summoned the question "how could these tasks be optimized?" and offered an invitation to tackle them mathematically.

The third trigger is the wish to pose a problem "here and now". It results from situations where an organizing committee of a forthcoming MC realizes that they need a problem in a particular content area and with a specific degree of difficulty to complete a problem set. Thus, the committee either turns to the EPPMC who specializes in the relevant content area or decides to construct such a problem on its own. Notably, the participating EPPMCs were unanimous in their dislike for such situations. As one of them declared, "The problem needs to come from being interested in something. The worst approach to posing a problem is to wish to pose it". Overall, the EPPMCs elaborated on the difficulty and lack of motivation to pose in such conditions, and five participants declared themselves incapable of coming up with problems in this way. Seven EPPMCs highlighted the low quality of problems that they posed "here and now".

JUXTAPOSITION

One can discern similarities between PP-requiring situations and the EPPMCs' "here and now" trigger. Indeed, mathematics education researchers often task students and teachers with free PP, where they are requested "to generate a problem from a given, contrived or a naturalistic situation" (Stoyanova & Ellerton, 1996, p. 519). This appears not very different from how an MC committee turns to EPPMCs to request problems in a particular mathematical domain and of a specific degree of difficulty.

Nevertheless, a difference needs to be highlighted regarding the number of requested problems. Within educational PP, it is not rare to ask learners to come up with multiple problems (e.g., Koichu, 2020; Silver et al., 1996; Silver & Cai, 1996) or even "as many problems as possible" (e.g., Kontorovich et al., 2012). However, MC committees mostly turn to a particular EPPMC for a single problem. This is not to say that EPPMCs generate a single problem while addressing this request. In Kontorovich and Koichu (2016), we presented a case of a single EPPMC – Leo – who posed "here and now" towards a forthcoming competition. In this process, he browsed through multiple mathematical ideas and classical problems from his memory before coming up with two new problems.

Speaking about the number of problems, at the very beginning of my study, I asked three EPPMCs to pose "several problems" based on the semi-structured Billiard task that has been used in previous PP research (e.g., Cifarelli & Cai, 2005; Koichu & Kontorovich, 2013; Kontorovich et al., 2012; Silver et al., 1996). All three rejected my requests categorically, suggesting that coming up with "several good problems is hard" and the Billiard task is "too hackneyed to come up with something new" (for the role of the feeling of innovation see Kontorovich & Koichu, 2012).

On the face of it, the a-didactical approach to educational PP (Koichu, 2020) resembles situations where EPPMCs extract mathematical phenomena from triggering mathematics and optimization-inviting real-life circumstances. The similarity is the status of PP as both the means for EPPMCs to learn new mathematics and optimize a solution of the real-life situation. However, the substantial difference is that the EPPMCs were the ones to initiate and abort PP. Indeed, the corresponding triggers can be better described with such constructs as EPPMCs reflecting-in-action (cf. Schön, 1983), being sensitive to opportunities to change a routinized practice (cf. Mason, 2002), and giving the space for PP as a habit of mind (cf. Cuoco et al., 1996). As Marion Walter put it, "[…] problem posing becomes second nature after you do it for a while. I seem to look at the world through 'problem posing' colored glasses" (Walter in Baxter & Walter, 1978, p. 122). I see reflections of these EPPMCs' triggers in the educational works of Cifarelli and Cai (2005), Contreras (2013), and Tall et al. (2017).

All in all, it appears to me that the differences overshadow the similarities between what triggers EPPMCs to engage in processes featuring PP and how mathematics educators often engage learners in this activity. These differences can be interpreted as PP-requiring situations being unfaithful to the authentic practices of some EPPMCs. But is this infidelity necessarily problematic? Next, I discuss the meaning of these differences for educational PP and sketch additional pathways to capitalize on the presented findings and specialist PP in general.

FROM SPECIALIST PP TO EDUCATIONAL PP

I contend that one should resist the temptation to label differences between specialist and educational practices as necessarily a problem of the latter. Specifically, I propose that three arguments should be in place before making this evaluative judgement. *First,* educators should be explicit about their PP goals. Recall that the initial motivation for looking at specialist PP was based on recurrent findings about the quality of problems posed by students and teachers. In other words, within this line of reasoning, generating quality problems has been construed as a goal. In turn, Cai and Leikin (2020) indicate that PP can be employed as a tool to pursue other instructional targets (e.g., deepening knowledge, developing mathematical creativity, advancing competencies). From this perspective, conceptual justifications are needed to consider whether learners' posing of "suboptimal problems" (cf. Cai et al., 2015) is necessarily problematic and whether there is any merit in bringing research on specialist PP to the table. *Second*, suppose educators desire that learners' PP should emulate specialist PP, at least to some extent. In that case, there is an argument for matching a particular community of learners to a specific community of specialists (cf. the problem of identifying the mathematical community in Weber et al., 2020). For instance, a priori, there seems to be nothing wrong about engaging pre-service school teachers in PP in a manner that is not consistent with what triggers problem posers for prestigious MCs. *Third*, given that empirical research on EPPMCs, and specialist PP in general, is in its infancy, it may be too early to discuss PP in community-wide terms (cf. the problem of heterogeneity in Weber et al., 2020). Realizing that single studies may not capture the practices of large and diverse communities begs the question of whether common features of educational PP should be criticized merely because they differ from what was found in single small-scale studies that do not necessarily represent broader specialist practices.

So how can research on specialist PP be of service for educational PP? As noted in the Introduction, this is a complex question, the debate on which the paper in-hand hopes to ignite. To make a first step, I briefly sketch three possible uses: (i) as *a body of relevant knowledge*, (ii) *as a lens for alternative interpretations of learners' PP*, and (iii) *as a trigger for new perspectives on and approaches to PP*.

First, this research could serve educators as a point of reference about how mathematical PP unfolds in different communities. This knowledge would inform educators about mathematics as a discipline and its uses, which they can "pass on" to their students. This research will be instrumental if educators decide to prepare their learners for authentic specialist PP or design analogous situations in their classrooms. For instance, Brown and Walter (1993) elaborate on a course in mathematics education where undergraduates and graduate students wrote articles for classroom journals and posed problems as part of these articles. This design can be construed as an educational version of mathematicians' PP, in which students acted as authors and critiques of the posed problems.

Second, research on specialist PP can provide educators with a different lens through which learners' PP activity can be considered. Let me illustrate this potency with EPPMCs. As I noted in the beginning, the problems posed by mathematics learners have been often described as simple, textbook-like, and not attractive even to learners themselves. The EPPMCs in my research used echoing descriptors concerning problems that they posed "here and now". Thus, educators' awareness of EPPMCs' practices may open the door to interpret the generation of "suboptimal problems" not as learners' deficiency but as an expert-like reaction to adverse PP-requiring situations.

Lastly, several recent publications have focused on fundamental definitions, conceptualizations, and implementations in the area of PP (e.g., Cai & Leikin, 2020; Liljedahl & Cai, 2021). In these publications, researchers take stock of the existing research in the area, often through identifying similarities and differences between previous works. The categories for capturing these differences and similarities might arise from the analysis of published studies (e.g., Baumanns & Rott, 2021; Papadopoulos et al., 2021). But ideas for categories can also come from outside of the area. In the spirit of Mason (2002), findings on specialist PP could serve as an external impulse that "triggers" new perspectives and illuminates familiar PP aspects in a new light. Learning about specialist PP might strike a chord with mathematics education researchers, and the fact that these practices unfold in a different context could cause a generative disturbance – "not a tidal wave, but a ripple sufficiently great to be distinguishable on the choppy surface which is my experience" (Mason, 2002, p. 68). For instance, the observation that learners are mostly engaged in PP through PP-requiring situations emerged from my research on EPPMCs, for whom it is only one scenario for generating problems. Insights into specialist PP beg the question of whether the identified aspects and perspectives may also play a role for learners, and how the area has attended to these up until now. Pursuing these questions may challenge existing conventions in the area, draw attention to studies less known due to their unconventionality, and lead to new ideas about how PP can be promoted among learners. In this way, research on mathematical practices of PP specialists may serve not as a beacon that points educational PP in the "right direction", but as a projector that sheds a distinct light on the landscape of endless possibilities.

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GROUP–1
Placing teachers at the center

A COMPARATIVE ANALYSIS OF FINNISH AND GERMAN PRIMARY GRADE TEACHERS' PROBLEM-SOLVING TEACHING PRACTICES

Ana Kuzle[1], Nina Sturm[2], Inga Gebel[1] and Anu Laine[3]

[1]University of Potsdam, Germany, [2]PH Ludwigsburg, Germany and [3]University of Helsinki, Finland

Problem solving is an indispensable process competence that is anchored in educational standards worldwide. Yet, according to somewhat weak empirical evidence, the level of implementation of problem solving in primary school mathematics is still considered unsatisfactory. In this study, we try to bridge this gap by analyzing 346 Finnish and German primary grade teachers' problem-solving teaching practices. For this purpose, we developed a questionnaire in which selected aspects of problem-solving implementation and influencing intrapersonal characteristics were considered. The results shed light on factors that may explain the unsatisfactory situation as well as the similarities and differences between the two countries.

INTRODUCTION

In recent decades, problem solving has become a central and indispensable educational goal of mathematics education worldwide due to its central role in later success in mathematics and in future work-life (Csapó & Funke, 2017; NCTM, 2000). According to both the Finnish National Core Curriculum for Basic Education (NBE, 2014) and the German National Standards for Primary Education (KMK, 2005), the purpose of mathematics instruction is not only to offer opportunities to learn mathematical concepts but also to develop problem-solving competence. Specifically, in the Finnish mathematics curriculum for Grades 1–6 (NBE, 2014), the goal of mathematics instruction is to guide students to develop their creative mathematical reasoning and problem-solving skills (Grades 1–6) as well as skills in posing questions and making reasoned conclusions based on their observations (Grades 3–6). In Germany at the end of Grade 4, students should be able to apply mathematical knowledge, skills and abilities to problem solving, develop and use problem-solving strategies, and recognize and use connections, and transfer them to similar situations (KMK, 2005). Thus, both countries place a value on developing and fostering problem-solving competencies from early grades on, with Finland being somewhat of a role model for the quality of mathematics teaching.

Despite these clearly formulated requirements, the teaching and learning of mathematical problem solving are still neglected in school practice, which has been exemplarily reported in the studies by Gebel and Kuzle (2019), Kuzle and

In I. Papadopoulos & N. Patsiala (Eds.), *Proceedings of the 22nd Conference on Problem Solving in Mathematics Education - ProMath 2022* (pp. 21–36). Faculty of Education, Aristotle University of Thessaloniki.

Gebel (2016), and Dreher et al. (2018) in the context of middle and secondary school mathematics. In this study, we aim to contribute to the field of mathematics problem solving by investigating the current state of problem-solving instruction in primary schools with regard to selected aspects of problem-solving implementation (i.e., frequency of implementation, type of implementation, obstacles, sources of teaching ideas) as well as some intrapersonal characteristics of teachers (i.e., professional background, teaching experience, attitude towards problem solving) that may be causal for this.

THEORETICAL BACKGROUND

Here, we present constraints that inhibit the implementation of problem solving from a general perspective as well as taking teachers' intrapersonal characteristics into consideration.

Problem-solving lessons in primary school between aspiration and reality in Finland and Germany

Even though there is a clear consensus that problem solving is indispensable even at the school level (NCTM, 2000), obstacles to its implementation in school mathematics are discussed at the level of the practitioners, lesson design, and school management (e.g., Herold-Blasius et al., 2019; Näveri et al., 2011; Pehkonen, 2017; Reiss & Törner, 2007), such as

- the marginal role of problem solving in teacher education (e.g., lack of subject-specific didactic knowledge, limited problem-solving experience, lack of meta-level reflection) (Kuzle & Rott, 2018; Reiss & Törner, 2007),
- general conditions and prerequisites (e.g., uncertainties in the planning of problem-solving lessons, unfamiliarity with teaching concepts, lack of (good) curricular materials) (Gebel & Kuzle, 2019; Herold-Blasius et al., 2019; Kuzle & Gebel, 2016; Pehkonen, 2017),
- faulty understanding of problem solving (e.g., word problems or puzzles) (Herold-Blasius et al., 2019; Näveri et al., 2011; Pehkonen, 2017),
- heterogeneity of the learning group (e.g., problem solving understood as an activity for high-achieving students, students' basic math skills and verbal abilities as obstacles) (Herold-Blasius et al., 2019; Reiss & Törner, 2007).

The list of identified constraints that inhibit the implementation of problem solving is far from complete. Moreover, this list should be viewed critically, as not only did it come from empirical studies with a weak empirical basis (e.g., small samples, author's teaching experience, earlier literature operating on a fragile database) but predominantly reported on studies with secondary grade teachers (e.g., Herold-Blasius et al., 2019). These results, however, may indicate that similar difficulties already exist at the primary school level.

Influence of teachers' intrapersonal characteristics on the implementation of problem-solving lessons

The teacher's role in the context of teaching-learning situations has a long tradition that is based on the search for and an investigation of influencing factors that can determine successful teaching, such as professional background, teaching experience, and teacher attitudes (Voss et al., 2015).

Professional background is a key potential predictor of teaching competence and teaching success with subject content knowledge, subject pedagogical content knowledge, and pedagogical/psychological knowledge as three core spheres of knowledge for teaching which can be examined from the perspective of qualification and teacher education (Baumert & Kunter, 2006). Whereas qualified (mathematics) teachers experience and acquire these three core spheres of knowledge for teaching during their teacher training program, this is not the case with non-qualified (mathematics) teachers who have usually not experienced the explicit teaching of two of the three spheres. Due to the lack of teachers in the German school system, KMK (2019) reported an increasing number of lateral entrants (13.3%) who have only been explicitly taught subject content knowledge and little or no subject pedagogical content knowledge or pedagogical/psychological knowledge. This is, however, not an issue in the Finnish educational system since teacher education, and especially primary teacher education, is a rather prestige choice of career. Only about 10% of applicants get accepted to primary teacher education at the University of Helsinki. The two countries also differ with respect to teacher education. Finnish primary teachers study education as their main subject in both bachelor's and master's studies. Specifically, they study didactics of all primary grade subjects; including 7 ECTS in mathematics didactics in their bachelor studies. Only 10–20% of the students at the University of Helsinki take the option of an additional mathematics module in their master's degree. Thus, most of the students study only 7 ECTS in mathematics didactics in their whole degree (from a total of 300 ECTS). Depending on the federal state, German primary teachers study two to three subjects. If mathematics is chosen, they receive credits in both their bachelor's and master's studies (e.g., in the federal state of Brandenburg 33 ECTS and 24 ECTS, respectively). Both perspectives raise the question of the possible effects of professional background on teaching quality regarding teachers' practices in the context of (primary grade) mathematics problem solving.

Teaching experience, as the main implicit factor in the acquisition of subject-specific knowledge, has been extensively studied as a potential predictor of teaching quality and student learning (e.g., Henry et al., 2011). Yet, research on the possible effects of teaching experience on teachers' practices in the context of (primary grade) mathematics problem-solving instruction is lacking.

Teachers' beliefs, practices, and *attitudes* are important for understanding and improving educational processes (Thompson, 1984; Voss et al., 2015). Herold-Blasius et al. (2019) reported that teachers' attitudes towards problem solving (e.g., not enough math is learned through problem solving, problem solving is only reserved for good students, problem solving is too difficult) influence their teaching practices. However, if and how different attitudes towards problem solving relate to concrete aspects of problem-solving implementation (i.e., frequency of implementation, type of implementation, obstacles, sources of teaching ideas) is unknown to our knowledge.

RESEARCH QUESTIONS

In view of the long-standing appreciation of problem-solving competencies in the curricula (e.g., KMK, 2005; NBE, 2014; NCTM, 2000), which, however, is still insufficiently reflected in school reality, the present study aimed to investigate specific problem-solving teaching practices and teachers' intrapersonal characteristics, and how they are connected, in both countries (Finland and Germany). Also, the similarities and differences between the two countries with respect to the above-mentioned aspects were investigated. The following research questions guided the study:

1. With consideration to the two countries, Finland and Germany,
    - how often is mathematical problem solving being taught?
    - how is problem solving being implemented?
    - what obstacles do the teachers report on?
    - which sources do the teachers use?
2. How are these aspects connected to the teachers' intrapersonal characteristics (i.e., teaching experience, professional background, teacher attitudes)?
3. To what extent is the effect of the above-mentioned intrapersonal characteristics on the specific problem-solving teaching practices moderated by the country (Finland and Germany)?

METHOD

Research design and subjects

For this study, a questionnaire-based mixed-methods cross-sectional study was chosen using a convenience sample. Here, primary schools were selected through existing contacts with the researchers' universities and through random inquiries. The study participation was voluntary. In total, 346 questionnaires were returned anonymously to the respective universities. The sample of 346 in-service primary grade teachers consisted of 160 Finnish teachers (Grades 1–6) and 186 German teachers (Grades 1–4). The German sample was collected in two federal states, namely Brandenburg and Baden-Württemberg (for more detail see Gebel et al., 2023). In terms of professional background, a total of 279 teachers taught mathematics as qualified teachers (80.6%) (n_F = 145, n_G = 134), and 61 as non-qualified teachers (17.6%) (n_F = 13, n_G = 48). The data of six persons were not provided. In terms of teaching experience, 46 teachers (13.3%) have been teaching mathematics for less than or up to two years (n_F = 21, n_G = 25), 60 (17.3%) for up to 5 years (n_F = 21, n_G = 39), 67 (19.4%) for up to 10 years (n_F = 34, n_G = 33), 75 (21.7%) for up to 20 years (n_F = 38, n_G = 37), and 96 (27.7%) for more than 20 years (n_F = 45, n_G = 51). The data of two persons were not provided.

Data collection instrument and data analysis procedures

For the study purposes, we used a questionnaire which was based on an adaptation of the instrument of Pehkonen (1993) and Kuzle (2017). Additionally, new items or statements for a specific item were developed or rephrased based on the literature on problem-solving instruction published in the last 20 years (e.g., Donaldson, 2011; Heinrich et al., 2015). The questionnaire consisted of three sections: (1) professional background and characteristics in teaching mathematics, (2) beliefs about problem solving, and (3) me and teaching problem solving (for more detail see Sturm et al., 2021). Each section consisted of several items with both open and closed questions. Here, section (1) of the questionnaire which consisted of 11 items is of relevance which – as stated above – focused on the teachers' actual problem-solving practices.

Concretely, in the context of the first research question, we examined four items, namely a) how often problem solving was implemented (frequency of implementation), b) in what ways the teacher implements mathematical problem solving in his or her instruction (type of implementation), c) what difficulties are associated with it (obstacles) and d) what sources are used in lesson planning (sources of teaching ideas). Regarding item a), teachers had to estimate on a five-point Likert-scale (1 = never, 2 = 1–2 times per school semester, 3 = monthly, 4 = weekly, 5 = almost daily) how often they integrated problem-solving activities into their mathematics lessons during the actual school year. Regarding items b)–d), teachers had to rate the extent to which they agreed with the given items on a five-point Likert-scale (1 = never, 2 = sometimes, 3 = occasionally, 4 = often, 5 = always) (see Table 1). Item b) was surveyed with 5 sub-items (e.g., whether problem-solving activities were incorporated into parts of lessons in mathematics), item c) with 9 sub-items (e.g., to what extent does lack of time make it difficult to plan and implement problem-solving lessons), and item d) with 10 sub-items (e.g., "Do you draw your ideas for problem-solving activities from textbooks?"). In order to answer the first research question and determine in which areas Finland and Germany differed significantly, the Mann-Whitney test was conducted. A t-test was not carried out, because the data were not normally distributed. Additionally, descriptive statistics were calculated.

In order to answer the second research question, we examined whether and to what extent the control variables 1) professional background, 2) teaching experience, and 3) attitude towards problem solving have an effect on the answers to items a)–d) for each country. The variable 1) professional background was differentiated into qualified teacher and non-qualified teacher. Variable 2) teaching experience was considered in three levels (1 = 0–5 years (less experienced teacher), 2 = 5–20 years[1] (experienced teacher), 3 = more than 20 years (more experienced teacher)). Variable 3) attitude towards problem solving was differentiated into positive, negative, ambivalent, and neutral (for more detail see Gebel et al., 2023) which revealed the following distribution: 129 positive (37.3%) (n_F = 68, 43.9%; n_G = 61, 37.0%), 41 negative (11.8%) (n_F = 23, 14.8%; n_G = 18, 10.9%), 61 ambivalent (17.6 %) (n_F = 30, 19.4%; n_G = 31, 18.8%), 89 neutral (25.7%) (n_F = 34, 21.9%; n_G = 55, 33.3%), and 5 not assignable. In order to determine how the aspects from research question 1 connected to the above-mentioned intrapersonal characteristics – thus, to answer the second research question – a descriptive statistics were conducted separately for both countries. Additionally, Fisher's exact test was used to examine inferentially whether significant relationships could be identified between the control variables and the various items. In contrast to the chi-square test, this test procedure can also be used to calculate smaller samples with table values smaller than five (Field, 2013).

In order to answer the third research question, we took the variable country as a dependent variable (see Figure 1). For this purpose, moderation regression analyses using product terms from mean-centered predictor variables (Hayes, 2018) were conducted with the dichotomous moderator country. These models imposed the constraint that any effects of the intrapersonal characteristics of teachers were independent of all other variables in the model.

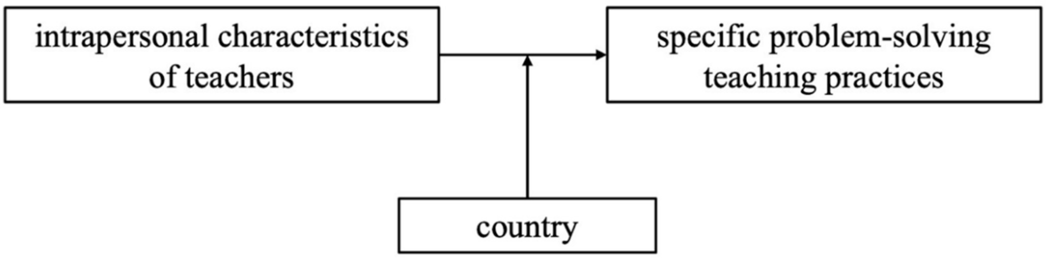

Figure 1. The moderation model with the single moderator country influencing the effect of teachers' intrapersonal characteristics on specific problem-solving teaching practices.

1. Although a finer subdivision was made in the questionnaire, the reported clustering was used for the purpose of a more homogeneous group distribution.

RESULTS

This section is divided into three parts and organized around the three research questions. Mainly results with significant differences are presented.

Problem-solving teaching practices in Finland and Germany

The analyses of specific problem-solving teaching practices (i.e., frequency of implementation, type of implementation, obstacles, sources of teaching ideas) revealed differences regarding all four designated areas between the two countries. Concretely, the frequency of implementation of problem-solving activities by Finnish teachers (*Mdn* = 4) is significantly higher than the frequency of implementation of problem-solving activities by German teachers (*Mdn* = 3) ($U = 16241.50$, $z = 2.93$, $p = .003$, $r = .16$).

In terms of how problem solving is implemented, the two countries differed significantly in whether they incorporate problem-solving activities into parts of lessons ($U = 18043.50$, $z = 4.47$, $p < .001$, $r = .24$), into whole math lessons ($U = 11307.50$, $z = -2.68$, $p = .007$, $r = -.15$), or into interdisciplinary projects ($U = 15858.50$, $z = 5.62$, $p < .001$, $r = .32$). Whilst the Finnish teachers include problem-solving activities more often in parts of lessons and interdisciplinary projects than the German teachers, the German teachers more frequently include problem-solving activities in whole lessons than the Finnish teachers.

In terms of obstacles, the two countries differed significantly only in the extent to which students' verbal abilities influence the planning and implementation of problem-solving lessons ($U = 10174.50$, $z = -4.48$, $p < .001$, $r = .24$). Students' verbal abilities are seen as an obstacle more strongly by the German teachers (*Mdn* = 3) than by the Finnish teachers (*Mdn* = 2).

The Finnish teachers (*Mdn* = 4) draw their problem-solving ideas significantly more often from textbooks ($U = 17460.00$, $z = 4.05$, $p < .001$, $r = .22$) or teacher's manuals (*Mdn* = 4) ($U = 20986.50$, $z = 8.11$, $p < .001$, $r = .44$) than the German teachers (in both cases: *Mdn* = 3). In contrast, the German teachers (*Mdn* = 2) more often look for inspiration in teacher journals ($U = 9931.00$, $z = -4.53$, $p < .001$, $r = .25$), and didactic books (*Mdn* = 3) ($U = 10893.00$, $z = -2.684$, $p = .007$, $r = .15$), and/or use ideas from education (university) (*Mdn* = 4) ($U = 9591.00$, $z = -4.237$, $p < .001$, $r = .24$) compared to the Finnish teachers (in all cases: *Mdn* = 2).

Influence of intrapersonal characteristics on problem-solving teaching practices

Using Fisher's exact tests, we examined the two countries separately to find whether there was a relationship between the above-mentioned intrapersonal characteristics and problem-solving teaching practices (see Table 1). The Finnish teachers' professional background has no significant effect on the sources of teaching ideas. However, significant influence of professional background can be observed in the German teachers, particularly in the sources they use for problem-solving lessons. Qualified teachers tend to be less likely to rely exclusively on textbooks or teacher's manuals than non-qualified teachers, whereas non-qualified teachers often do not or rarely depend on professional development or on their training period (i.e., ideas from education).

	PB		**TE**		**TA**	
	p_F	p_G	p_F	p_G	p_F	p_G
Frequency of implementation				.004	<.001	
Type of implementation						
part of lessons				<.001	.017	
whole lessons		.002	.002	.006	.002	.045
integrated into weekly plans					.002	
integrated in interdisciplinary projects				.001	.018	
several successive hours					.018	
Obstacles						
lack of time						
lack of good tasks		.001			.029	
heterogeneity					.030	
basic math skills					.003	.018
verbal abilities					.035	
students' motivation					.003	.009
lack of self-regulation skills						
own problem-solving competence					.001	
own attitude to problem solving					.001	.005
Sources of teaching ideas						
textbook		.023				
teacher's manual		.003				
teacher journals						
professional development		<.001				.027

didactic books			
ideas from education (university)	.018	.005	<.001
own spontaneous ideas	.038		
current occasions and spontaneous questions of students			.008
conversations with colleagues			.003
websites		.009	

Note. PB = professional background; TE = teaching experience; TA = teacher attitude; F = Finland; G = Germany.

Table 1. Results of Fisher's exact test for the relationship between specific problem-solving teaching practices and teachers' intrapersonal characteristics.

In both Germany and Finland, the teachers' teaching experience influences their problem-solving teaching practices (i.e., implementation and sources) which is stronger in Germany than in Finland. Descriptive statistics revealed that more experienced German teachers implement problem-solving activities in their daily or weekly lessons significantly more often than less experienced and experienced teachers. Furthermore, 70% of experienced or more experienced teachers never or only rarely use problem solving in parts of lessons, whereas 80% of them never or only rarely integrate problem solving in interdisciplinary projects. Also, there are significant differences in both Finland and Germany regarding the use of ideas from education (university). The fact that more experienced teachers never or rarely refer back to their university period (Finland: about 75%, Germany: about 70%), and less experienced teachers often or always refer back to their university period (in both countries nearly 50%) can be observed comparably in both countries.

Significant effects of attitudes towards problem solving in the areas of frequency of implementation, type of implementation, and obstacles occur more frequently among Finnish than among German teachers. Regarding the frequency of implementation, Finnish teachers with a positive attitude towards problem solving use problem-solving activities more often than teachers with a negative attitude. This cannot be transferred to Germany, since teachers with a negative attitude integrate problem-solving activities on a monthly, weekly or almost daily basis comparable to the other groups, and, therefore no significant differences between the groups of teachers exist. For all subitems of the types of implementation, descriptive statistics revealed that Finnish teachers with a positive attitude integrate problem-solving activities into parts of lessons, whole lessons, weekly plans, interdisciplinary projects, or several consecutive hours more often than teachers with a negative attitude. These teachers never or seldomly integrate problem-solving activities. Regarding obstacles, the descriptive statistics revealed that Finnish teachers with a negative attitude tend to rate the occurrence of all obstacles as high or very high, whereas teachers with a positive attitude perceived these obstacles as less important as these never or rarely influenced their planning and implementation which was not the case with German teachers.

Differences between Finland and Germany regarding problem-solving teaching practices and intrapersonal characteristics

The moderation regression analysis with the intrapersonal characteristic professional background revealed a significant main effect of professional background and a significant moderator effect of the country on the professional background as a predictor of frequency of implementation. The main effect of the professional background indicated that qualified teachers had a higher frequency of implementation. Whereas qualified teachers of both countries hardly differ regarding frequency of implementation, German non-qualified teachers integrate fewer problem-solving activities into their lessons than qualified teachers. For Finnish teachers, it is the exact opposite: non-qualified teachers integrate problem-solving activities into their lessons more often than qualified teachers.

The influence of teaching experience on specific problem-solving teaching practices is moderated by the country in three moderation models (i.e., obstacles due to students' lack of mathematical skills, verbal abilities, and motivation). No significant main effects exist in any of the three models. Regarding the significant moderation effects, German teachers attribute a high influence to students' lack of basic math skills, independent of their teaching experience. The Finnish less experienced teachers attribute more importance to this obstacle than the Finnish more experienced teachers. Students' verbal abilities hinder both Finnish and German teachers from integrating problem-solving activities into their lessons. This becomes even more important for the more experienced German teachers, and is of less importance for the more experienced Finnish teachers. Students' lack of motivation is seen as an obstacle for both German and Finnish teachers with hardly any country difference among less experienced teachers. But, the more experienced German teachers attribute higher importance to this obstacle, which is considered of less importance by the more experienced Finnish teachers.

Regarding the intrapersonal characteristic teacher attitudes towards problem solving, there was a significant moderation effect in one model only. The obstacle lack of students' math skills is rated high by the German teachers independent of their attitude. Whereas the Finnish teachers with a negative attitude viewed this obstacle as important, the Finnish teachers with a positive attitude considered it less important.

DISCUSSION

In the last section, a differentiated picture on specific problem-solving teaching practices in primary schools in Finland and Germany is discussed, the study limitations are considered, and some possible future research directions are given.

Finnish and German primary grade teachers' problem-solving practices

Frequency of implementation. Generally, Finnish teachers integrate problem-solving activities more often into their mathematics lessons than German teachers. It is intriguing that the more experienced German teachers integrate problem solving into their lessons more often than the less experienced teachers. It may be that experienced teachers have the knowledge (and courage) to teach problem solving precisely because of their many years of experience, that they are convinced of its importance, and that they are willing to spend time on it. This effect does not apply to the Finnish sample. On the one hand, it may be that all Finnish teachers – regardless of their teaching experience – take the curriculum requirements seriously, and comply with them. On the other hand, the Finnish teachers' attitudes towards problem solving significantly influence the frequency of implementation. Teachers with a positive attitude towards problem-solving implement problem-solving activities more often in their lessons than teachers with a negative attitude towards problem solving. This is consistent with findings that teacher attitudes enhance educational processes (Voss et al., 2015), and attitudes towards problem-solving influence their teaching practices (e.g., Herold-Blasius et al., 2019). However, this effect could not be replicated in the German sample. Thus, it can be concluded that the frequency of implementation among Finnish teachers is influenced by their attitudes towards problem solving, and among German teachers by their teaching experience.

Type of implementation. With regard to the type of implementation, the Finnish teachers include problem-solving activities more as parts of lessons and in interdisciplinary projects than the German teachers who tend to use whole lessons, but rather less frequently. That Finnish teachers integrate problem solving more often in interdisciplinary projects may be due to the fact that interdisciplinary projects are explicitly anchored in the curriculum (NBE, 2014) and, thus required which is not the case in Germany. In terms of their implementation practices, descriptive statistics revealed that Finnish teachers with a positive attitude towards problem solving differ from teachers with a negative attitude which extends the works of Herold-Blasius et al. (2019), and Voss et al. (2015). This was found only once in the German sample regarding implementation in whole lessons. In contrast, the teaching experience of German teachers influences the type of implementation in more subitems than was the case with Finnish teachers. For example, experienced German teachers focus on integrating problem solving only occasionally in individual lessons, whereas experienced Finnish teachers do this often or always. This may be due to Finnish teacher's manuals that include problem-solving tasks for every lesson (Hemmi et al., 2018) which is not the case in Germany. Regarding the influence of professional background, only one influence on the type of implementation was found. Qualified teachers focused on integrating problem solving into whole lessons, whereas non-qualified teachers did this rarely.

Obstacles. The results regarding the obstacles to implementing problem solving do not present a unified picture, especially when taking intrapersonal characteristics into consideration. For instance, German teachers often consider students' verbal abilities as an obstacle which confirmed the existing results (e.g., Herold-Blasius et al., 2019; Reiss & Törner, 2007). This was not confirmed with respect to basic math skills; it was only seen in Finnish teachers who had a negative attitude towards problem solving. Here, it would be interesting to know what understanding of problem solving and problem-solving activities these teachers or teachers in general have. It may be that they are referring to solving word problems or puzzles rather than inner-mathematical problems, as has already been reported (e.g., Herold-Blasius et al., 2019; Näveri et al., 2011; Pehkonen, 2017). Any further significant influence of teachers' attitudes to obstacles was primarily observed in the Finnish sample. Finnish teachers with a positive attitude towards problem solving rate the obstacles as less significant than the teachers with a negative attitude. This was only found in a few characteristics in the German sample. It may be that teachers who view problem solving positively are aware of the influences but are able to address these obstacles appropriately to facilitate problem solving activities for all students. It is also possible that they had only had positive experiences with the implementation of problem-solving activities and therefore did not make any negative connections with the obstacles mentioned. This positive or negative view of the obstacles could in turn be the reason for the frequency or infrequency of the implementation of problem-solving activities.

Sources. Unlike German teachers, Finnish teachers most often use the textbook and teacher's manual as a resource. German teachers make more use of teacher journals, didactic books, and ideas from their education (university) (for more detail see Gebel et al., 2023). This may be due to the fact that textbooks in Germany contain fewer problem-solving tasks (or often within additional tasks), and that teaching is organized more subject-specifically (Gebel & Kuzle, 2019; Kuzle & Gebel, 2016) whereas in Finland textbooks are of quite a high quality and teacher's manuals usually include problem-solving tasks for every lesson (Hemmi et al., 2018). Additionally, among German teachers referring to the textbook and teacher's manual is influenced by their professional background. Reasons for this can be manifold. On the one hand, non-qualified teachers may lack alternative teaching materials, in their education problem solving was dealt with only to a limited extent, if at all, they do not attend further professional development or they prefer to attend further professional development in other subjects or other mathematical areas. On the other hand, it may be that they lack some aspects of the three core knowledge spheres making it difficult to select appropriate sources (Baumert & Kunter, 2006). The influence of professional background could not be observed in Finland. This may be due to the fact that Finland rarely employs non-qualified teachers. Regarding their teaching experience, Finnish and German teachers differ in terms of recourse to their university period. Unsurprisingly, more experienced teachers never or less often revert to their time at university than teachers who are more recently qualified. If they were at university more than twenty years ago, it is likely that teachers will not remember or have materials from that period and that problem solving was not part of their studies. Furthermore, it should be taken into consideration that only a marginal role in teacher education was attributed to problem solving at that time (Kuzle & Rott, 2018; Reiss & Törner, 2007).

To conclude, the study results expand the influence of the teacher's intrapersonal characteristics (i.e., professional background, teaching experience, teacher attitude) on successful teaching (Voss et al., 2015) to the problem-solving teaching practices of primary school teachers.

Limitations of the study and future research directions

This study was a mixed-methods study using convenience sampling. Thus, the participating teachers only represented the two countries to a limited extent. Also, due to voluntary participation, it may be assumed that the teachers were motivated and that the results reflect the practices of motivated teachers. However, this is uncertain since the data did not reveal in all cases a unified picture. Furthermore, the results may be limited to specific cultural characteristics of both countries. For the generalizability of the results in a wider setting, it is essential to recruit a larger sample from a variety of settings (e.g., federal states, counties or countries) using alternative sampling strategies (e.g., maximum variation sampling, probability sampling) as well as to triangulate the data (e.g., classroom observation), so that the researcher could create a less biased and more thorough description of the current state of problem-solving instruction on both a national and international level.

The results have also provided evidence of possible theoretical as well as methodological biases. With respect to the former, an extensive literature review concerning obstacles to implementing problem solving was undertaken to develop the questionnaire items. However, it cannot be said that all aspects were found, especially since the published work mainly reported on secondary education, and the methodological basis in the found literature was somewhat weak. With respect to the latter, the structure of the questionnaire was not optimal: some items were not entirely or fully answered by all participants, some inaccuracies were evident in the translation (e.g., more technical language was used in the German version than in the Finnish version of the questionnaire), no data triangulation was used, and the teachers' understanding of the "problem solving" and "problem-solving activities" constructs was questionable (e.g., word problems, difficult tasks). A re-design of the questionnaire regarding both of these aspects is, thus, another possible research direction. Last but not least, even though the results of the study confirm some experiences from practice-based contributions (Herold-Blasius et al., 2019) from the secondary school sector, the study results revealed that further research is needed to shed more light on problem solving in primary schools, and to address the open questions regarding possible obstacles to its implementation in the mathematics classroom.

CONCLUSIONS

It is clear that – and similar to findings of earlier research studies (e.g., Dreher et al., 2018; Gebel & Kuzle, 2019; Gebel et al., 2023; Kuzle & Gebel, 2016) – problem solving in the primary school classroom needs to be improved. The role of teachers in this matter is paramount. The study showed that 54.9% of the teachers expressed positive views about problem-solving instruction (37.3% fundamentally positive, 17.6% ambivalent), and also partially recognized its didactic potential. Adequate and solid teacher training focusing on both content and pedagogical content knowledge related to problem solving as well as on developing positive attitudes towards problem solving should not be underestimated but is essential as is supplementary training. This may dispel uncertainties regarding problem-solving instruction and allow for its regular implementation in school mathematics. Also, the value of accessible, high-quality, and differentiated sources for teaching problem solving should not be underestimated as this may help in building a bridge to overcome the reported obstacles. Continuous empirical studies on the status of problem-solving instruction at different educational levels (e.g., school, university) as well as how teachers could be supported in implementing optimal problem-solving lessons and at the same time be supported in conducting high-quality problem-solving instruction need to be conducted to keep up to date with developments. Only in this way can general statements be substantiated, reality be depicted, and issues and obstacles be resolved.

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DISCOVERING SOME DIFFICULTIES OF PRE-SERVICE PRIMARY TEACHERS IN A PROBLEM-SOLVING PROCESS BY USE OF GEOMETRIC FIGURES

Branka Antunović-Piton[1] and Nives Baranović[2]

[1]Faculty of Education, University of Ljubljana, Slovenia

[2]Faculty of Humanities and Social Sciences, University of Split, Croatia

This article presents the pilot study results on 75 Croatian pre-service teachers' difficulties on solving geometric problems. Particularly, we discuss visualization processes as possible difficulties causes based on Duval's framework of geometrical figure apprehension. A mixed collated data analysis reveals that pre-service teachers use all types of geometrical figure apprehension, but not mutually and almost incorrectly due to poor visualization skills, misunderstandings or misapplication of geometric concepts. These results indicate the need to systematically include more visualization elements into geometry teaching at all educational levels, starting with pre-service teachers.

INTRODUCTION

Based on many teachers' personal experiences and educational research findings, it has been known that students at all levels struggle to work with geometrical concepts and solve geometric problems (e.g., Baranović & Antunović-Piton, 2019; De Villiers, 2010; Duval, 2017; Fujita & Jones, 2007). The key elements in working with geometrical concepts are geometric figures, which causes many difficulties due to their dual nature. On one hand, geometric figures are abstract geometrical concepts, and visual representations that illustrate these concepts on the other hand (Fischbein & Nachlieli, 1998). For this reason, it is important to recognize not only how geometric figures are shown but also what they represent. In order to overcome this transition from a concrete representation to an abstract concept, which is called the geometric eye development (Fujita & Jones, 2002), appropriate visualization skills are needed (Duval, 2006). Possible causes of difficulties in understanding and applying geometrical concepts are insufficient development of visual-spatial abilities and skills of coordinating different processes in working with geometric figures (Fujita & Jones, 2007; Presmeg, 2014).

If teachers are unaware of the difficulties in learning geometry, they cannot teach their students how to overcome these problems. Thus, pre-service teachers must

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first develop their own geometric eye so they will be able to analyse and cope with students' difficulties in geometry (Presmeg, 2006). Therefore, the aim of the pilot research, as part of a larger action research on the development of geometric thinking, was to examine which visualization processes pre-service teachers use when working with geometric figures trying to solve geometric problems, and to discover possible difficulties that hinder these processes. With this aim, appropriate geometric problems based on elementary geometrical concepts were selected.

THEORETICAL BACKGROUND

Geometry is a natural environment for the application of visualization, and it is undeniable that poor visualization skills can cause many difficulties as regards learning geometrical concepts along with understanding and applying these concepts in solving problems. In mathematics education, the definition of visualization (Presmeg, 2014) presented by Arcavi is widely accepted:

> "Visualization is the ability, the process, and the product of creation, interpretation, use of, and reflection on upon pictures, images, and diagrams, in our minds, on paper or with technological tools, to depict and communicate information, thinking about, and developing previously unknown ideas and advancing understandings." (Arcavi, 2003, p.217):

According to this definition, visualization means ability, process and product, and the potential of visualization is achieved through systematic learning and teaching (Presmeg, 2006). However, visualization processes and the transition between multiple representations are neither linear nor straightforward and are very cognitively demanding. Consequently, these processes are refused by almost every student and many teachers. Moreover, "For a mathematician and a teacher, there is no real difference between visual representations and visualization. But for students, there is […]. They do not see what the teacher sees or believe they will see." (Duval, 2014b, p.160).

For successful work with geometric figures in order to solve geometric problems, Duval (1995) proposes a theoretical framework in which he distinguishes four geometrical figure apprehensions (GFA): perceptual, sequential, discursive, and operative apprehension. In addition, *geometrical figure* (GF) stands as a type of visual representation with a heuristic role that is necessary for problem-solving or proving.

Working with geometric figures have to begin with perceptual processing and continue with mathematical processing in order to discover the mathematical message. Perceptive processing of GF is the result of unconscious activities that vary from person to person and depend on their knowledge and experiences, which achieves perceptual understanding. So, *perceptual apprehension* (PA) refers to what figures and subfigures a person recognizes at first glance and can name. In other words, PA of GF means what GF shows.

Mathematical processing of GF is the result of conscious activities and knowledge, and depending on the processing method, it can be sequential, discursive or operative, thus achieving sequential, discursive or operative GFA. *Sequential apprehension* (SA) refers to every GF construction or reconstruction process and describes these processes by critical steps. The construction process means recognizing and understanding the critical steps of the GF creation process, while reconstruction means discovering these steps of constructed GF according to the mathematical properties. The critical steps mean all basic and elementary constructions into which some complex structures can be decomposed. Hence, SA depends on the technical constraints of the creation process and mathematical properties embodied in basic and elementary constructions.

Discursive apprehension (DA) refers to the mathematical properties embodied in GF, that is, establishing a relationship between the elements of the GF and mathematical properties through definitions, axioms, and theorems. *Operative apprehension* (OA) refers to modifying a given GF to get an insight into the solution of the geometric problem or the key idea to prove the statement. Therefore, OA means the heuristic use of GF in problem-solving or proving. According to Duval (1995, p.123), modification of GF is based on three methods: (1) *reconfiguration method* - splitting GF into pieces and moving pieces like puzzle pieces to form a new figure; (2) *supplement method* - supplementing new elements in given GF to gain a deeper insight; (3) *transformation methods* - changing GF's shape, size, or orientation to find out a new perspective. So, mathematical processing of GF means discovering what GF represents, it is realized by at least one of SA, DA or OA, and PA is the first and necessary step (Duval, 1995, p.147).

Various educational studies confirm the usefulness of Duval's theoretical framework for working with geometric figures in problem-solving (Gridos et al., 2022; Michael et al., 2011; Panaura & Gagastis, 2010). More precisely, constructing knowledge of geometrical concepts and solving geometric problems or proving statements is the essential interplay between seeing and saying, visualization, symbolism, and language for stating and deducing properties. Therefore, in this research, the described Duval's theoretical framework is used to analyse the process that pre-service teachers use when solving geometric problems.

AIMS AND RESEARCH QUESTIONS

Considering the observed problem and the importance of visualization processes in working with geometric figures, the main purpose of this research is to gain insight into these processes of pre-service teachers, particularly when solving selected geometric problems according to Duval's theoretical framework.

Therefore, the aim was not only to examine the extent to which pre-service teachers successfully solve the given tasks, but also to gain insight into the strategies used and to identify possible difficulties and factors that enable or hinder the process of problem-solving. The selected tasks include several basic geometrical concepts of elementary school mathematics, and according to their requirements as they belong to tasks with higher cognitive demands (Smit & Stein, 1998; Hsu, 2013).

In order to examine the set aims, we put the following research questions:

1. To what extent are pre-service teachers successfully solving given geometrical tasks?
2. What impedes the problem-solving process of pre-service teachers?
3. What types of geometrical figure apprehension whereby pre-service teachers mostly use during geometrical problem-solving?

METHODOLOGY

Descriptive and quantitative analyses were mutually used to obtain results from the tasks completed by pre-service teachers. Namely, by using mixed analysis, research questions can be answered more completely than when qualitative and quantitative analyses are used separately (Lund, 2012).

Sample

The pilot study reported here is part of action research in which we explore students' geometrical thinking development. The focus was on 75 pre-service teachers, 19-20-year-old students from the second and third year of Faculty for primary education teachers in Croatia. Among these participants, 54.67% (41 out of 75) of them is from the second year, and 45.33% (34 out of 75) them from the third year.

The reason for choosing this sample is to gain insight into the degree of pre-service teachers' geometric eye development, before learning geometry in the second year and after learning part of geometry in the third year. Because, if pre-service teachers do not have their own geometric eye sufficiently developed, they will not be able to cope with the difficulties in learning and teaching geometry.

Personal information about participants was not requested. The participation was voluntary and anonymous - each of the participants was assigned a unique ID code, following ethical research practice (Cohen et al., 2007).

Instrument

This instrument consists of four tasks, each containing a geometric figure and the text (see Appendix). The tasks were formed according to the Duval framework of geometrical figure apprehension (Duval, 1995), and similar tasks were used in other research on visualization processes (e.g., Fujita & Jones, 2002; Michael-Chrysanthou & Gagatsis, 2013).

In Task 1 (Counting triangles) it is required to count all triangles in a complex figure and name them using highlighted points (vertices of triangles). In Task 2 (Area of a triangle) it is necessary to determine the area of an obtuse triangle inside a rectangle placed in a square grid. In Task 3 (Creating new figures) all possible figures should be formed from two given congruent scalene right-angled triangles, and the types of formed figures should be determined. In Task 4 (Perimeter of figures) it is required to compare two figures in a square grid according to their perimeters.

Furthermore, in order to successfully solve the geometrical problem, in all four tasks the given geometrical figures have to be broken down into figural units of the same or lower dimension units that figures are composed of (PA, SA and OA). After these deconstructions, it is necessary to make links between figural units through the geometrical properties (PA, SA, OA and DA) to get the mathematical message. Considering that each task can be solved in several ways, the success of the chosen strategy depends on dominating individual visualization processes as well as working with GF (Leikin, 2010; Gagastis & Geitona, 2021). Therefore, an explanation of the solution procedure is requested in all tasks to gain insight into the used strategies and visualization processes.

Data collection and analysis

We conducted the pilot study in September 2021. The participants had no preparation and they had 20 minutes to solve tasks at the beginning of the class session. Given that there are two tasks with similar outcomes (Task 1 and Task 3 of one type and Task 2 and Task 4 of another type), two pairs of tasks with different outcomes were formed: (Task 1, Task 2) and (Task 3, Task 4). All participants in the class were randomly divided into two groups and each participant solved one task pair. Finally, 49.33% (37 out of 75) of participants solved the first task pair and 50.67% (38 out of 75) of them solved the second task pair.

After collecting and reviewing written participants' papers, assessment criteria were defined by the two researchers, first separately and then collectively according to the research question. Quantitative analysis of the participants' answers gave an insight into the success of solving given problems, and qualitative analysis of the answers comparatively provided an insight into their visualization processes, strategies, and other factors that influenced the problem-solving success.

RESULTS AND DISCUSSION

Analysis and discussion of results have been carried out based only on research questions, from the aspect of success in geometrical problem-solving and the aspect of GFA processes. Namely, this paper does not compare the performance between participants of different years or between tasks, but analyses the results for each task separately, with an emphasis on the strategies and visualization processes used in working with GF. Finally, a discussion of all presented results is provided.

Task 1 - Counting triangles

Table 1 shows the distribution of success in counting triangles within a given geometric figure. Although similar tasks are used in mathematics classes, the participants' performance differs from expectations.

Number of triangles	N	%
11	14	37.84
10	12	32.43
9,8,7	8	21.62
0	3	8.11
Sum	37	100.00

Table 1. Success in counting triangles.

According to the results (see Table 1), only slightly more than a third of the participants (14 out of 37; 37.84%) successfully counted all the triangles in a given geometric figure, and among them, some listed some triangles twice. Then, less than a third of the participants (12 out of 37; 32.43%) omitted one triangle when counting, and there were also doubled triangles. Finally, slightly less than a third of the participants (11 out of 37; 29.73%) were not successful in counting all the triangles: eight of them (8 out of 37; 21.62%) omitted 2, 3, or 4 triangles while three of them (3 out of 37; 8.11%) stated only an incomplete number, but not the name of triangles.

Through a qualitative analysis of the participants' works, it is evident that only these participants who counted the triangles systematically with a specific strategy were successful in counting, while the others did not find all the triangles. The most frequently omitted triangles were the triangles AEC (11 out of 37; 29.73%), ABF (9 out of 37; 24.32%), and ABE (5 out of 37; 13.51%), i.e., those triangles which are composed of several parts.

In the counting process, two types of strategies dominated: listing the triangles in order according to the vertices, starting from A, B, or C, and listing the triangles from the smaller inside to the outside or vice versa, from the largest outside to the inside. Some counted from left to right, others from right to left or alternately.

Also, slightly more than half of the participants consider the orientation when naming the triangle, while the others do not, regardless of the strategy used.

According to Duval's GFA, it can be said that the participants successfully used PA because almost all of them listed the triangles seen "at first sight". Regarding the participants who listed triangles composed of two or more parts, it can be said that they have the skill of changing attention from the elements of a geometric figure to the whole and vice versa, i.e., they have a developed ability to see the figure in different ways, what is a feature of OA. Furthermore, most participants listed triangles only once, regardless of orientation, which means that they know the concept of a triangle as a figure defined by three non-collinear points, which is a feature of DA. The skill of listing triangles using a specific strategy is a feature of SA.

Task 2 - Area of a triangle

Table 2 shows a distribution of success in determining the area of an obtuse triangle within a rectangle a given area included and placed in a square grid. The participants' works were evaluated according to three criteria: correct result and explanation (T), incomplete or incorrect result and explanation (F), and no answer (NO).

Area of triangle	N	%
T	5	13.51
F	27	72.97
NO	5	13.51
Sum	37	100

Table 2. Success in computing the area of a triangle.

According to the results (see Table 2), almost a seventh of the participants (5 out of 37; 13.51%) did not give any answer, and the same number of participants were successful in determining the required area, while over 70% (27 out of 37; 72.97%) had significant difficulty in determining the way to a solution. In particular, many participants (22 out of 37; 59.46%) did not complete the process. Over 60% of the participants (24 out of 37; 64.86%) determined the lengths of the sides of the rectangle, and most of them used the square grid and the area formula ($P = ab$, $a = 6$, $b = 4$). Considering the fact that they recognized right triangles inside the rectangle, and it is evident that there is almost half of the participants (17 out of 37; 45.95%) used Pythagoras' theorem.

Qualitative analysis of the solution process reveals three strategies: (1) direct computation using the formula, (2) indirect computation by subtracting the area of triangle EBC from the area of triangle ABC and (3) splitting triangle AEC into two smaller triangles and adding their areas. All participants with the correct solution used the strategy of indirect area computation. In contrast, the remaining two strategies did not successfully lead to the correct solution, and the reasons are numerous. Thus, some participants used the wrong formulas for the area of a triangle (e.g., *P = abc)* or applied Pythagoras' theorem to an obtuse triangle. As well, the participants unsuccessfully determined the triangle altitude, because they read its length from the grid, not based on the properties, but according to how it looks. None of the participant established a direct connection between triangle AEC and triangle BCE using the area method, and the fact that the area of the required triangle corresponds to a quarter of the area of the given rectangle.

Deducing lengths based only on the image (how it looks) indicates the dominance of PA over logical reasoning, i.e., over the correct application of DA.

Determining the lengths of the rectangle sides by the square grid in which it is placed and combining it with the formula indicates the mutual use of SA and DA. However, the measures that participants could not determine through the appropriate formula were determined based on the image (how it looks). That means there is the absence of DA (e.g., determining the length of the altitude) or incorrect use of DA (e.g., application of Pythagoras' theorem to an obtuse triangle). PA becomes dominant. To establish connections between triangles, it is necessary to operate using the figure and apply the appropriate formula, i.e., both of OA and DA mutually, which is successfully used by only a small number of participants.

Task 3 - Creating new figures

Table 3 shows the distribution of success in creating all possible figures from two congruent scalene right-angled triangles.

According to the data presented (see Table 3), a visibly high percentage of participants drew the first three figures, in contrast to the last three, which were made by a much smaller number of participants. Thus, the isosceles triangle was drawn by all participants except one (97.37%). The second isosceles triangle and rectangle were drawn by more than 80% of them. In comparison, about a sixth (15.79%) of participants drew the first parallelogram and deltoid, while a fifth (21.05%) of them drew the second parallelogram.

Looking at the overall works by participants, only three (7.89%) drew all six figures. Besides them, 23 participants (60.53%) drew the first three figures, and 12 participants (31.58%) had different combinations of drawn figures (from 1 to 5). The participants who successfully created all the figures used strategic matching, while the others did not.

Naming figures were made by 30 participants (78.95%) and were quite diverse. Namely, some participants listed only classes (triangle, quadrilateral), and some listed specific figures as isosceles triangles, a rectangle, parallelograms, and a deltoid. However, some participants misnamed the type of figure, e.g., equilateral triangle, square, or rhombus.

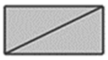
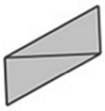
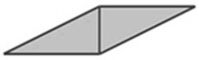
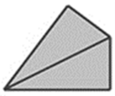
Visual representation	Type	N	%
	Isosceles triangle 1	37	97.37
	Isosceles triangle 2	32	84.21
	Rectangular	34	89.47
	Parallelogram 1	6	15.79
	Parallelogram 2	8	21.05
	Deltoid	6	15.79

Table 3. Success in creating figures.

According to the data presented (see Table 3), a visibly high percentage of participants drew the first three figures, in contrast to the last three, which were made by a much smaller number of participants. Thus, the isosceles triangle was drawn by all participants except one (97.37%). The second isosceles triangle and rectangle were drawn by more than 80% of them. In comparison, about a sixth (15.79%) of participants drew the first parallelogram and deltoid, while a fifth (21.05%) of them drew the second parallelogram.

Looking at the overall works by participants, only three (7.89%) drew all six figures. Besides them, 23 participants (60.53%) drew the first three figures, and 12 participants (31.58%) had different combinations of drawn figures (from 1 to 5). The participants who successfully created all the figures used strategic matching, while the others did not.

Naming figures were made by 30 participants (78.95%) and were quite diverse. Namely, some participants listed only classes (triangle, quadrilateral), and some listed specific figures as isosceles triangles, a rectangle, parallelograms, and a deltoid. However, some participants misnamed the type of figure, e.g., equilateral triangle, square, or rhombus.

A minor number of participants (5 out of 38; 13.16%) highlighted the right angle and marked the side lengths, indicating that they created figures more based on appearance (PA) and less on properties (DA). The requirement of the task to form new figures from the given figures imposes the need to operate with the figure (move, turn, flip), which requires the skill of operational figure processing (OA), and the examination of all possibilities requires strategic matching (SA). However, the participants showed neither the skill of operating with figures, nor the strategy of examining all possibilities, which resulted in a very poor performance.

Task 4 – Perimeter of figures

Table 4 shows the distribution of the success in comparing two geometric figures according to their perimeter for questions A and B. In question A, the distribution is shown according to the answers offered. In question B, the explanation was evaluated according to three criteria: correct explanation (T), incomplete or incorrect explanation (F), and no explanation (NO).

Item A	N	%
1 (True)	14	36.84
2	2	5.26
3	20	52.63
4	1	2.63
NO	1	2.63
Sum	38	100.00

Item B	N	%
T	6	15.79
F	25	65.79
NO	7	18.42
Sum	38	100.00

Table 4. Success in comparing figures by perimeter.

According to the results (see Table 4), the correct answer in question A was given by slightly more than a third of the participants (14 out of 38; 36.86%), and between them, the correct explanation in question B was given by slightly less than half of them (6 out of 14; 42.86%). That means that the task was completed correctly by less than a sixth of the participants (6 out of 38; 15.79%), which is a relatively poor result considering that the concept of the perimeter was used continuously throughout the educational period before the tertiary level.

Analysing of the participants' works indicating the dominance of one strategy (15 out of 38; 39.47%): translating the corresponding lengths into a numerical value, using formulas to determine the perimeter of a rectangle and circle, and adding or subtracting the obtained perimeters. However, only two participants (2 out of 38; 5.26%) successfully computed and compared the perimeters of the given figures using this strategy.

The remaining 4 participants (4 out of 38; 10.53%) who successfully compared the perimeters of the figures used the strategy of comparing the parts of the figures' edge. The rest of the participants used the other two strategies: counting unit squares (9 out of 38; 23.68%) or comparing the halves of the given figures (4 out of 38; 10.53%), making the wrong inference about the equality of perimeters. In other words, more than a third of the participants (13 out of 38; 34.21%) confused the concept of perimeter and area.

To determine the lengths of the sides of the rectangle and the arcs of the circle or rectangle area, the participants used the grid in which the figure was placed and set the unit measures for the lengths or area, which indicates the use of elements of SA and PA. Furthermore, to compare the perimeters of the given figures, some of the participants computed the length of the edge of the given figures by applying the formula for the perimeter of a rectangle as well as the formula for the perimeter of a circle, while some of the participants computed the area of the figures by applying the formula for the area of a rectangle, which indicates DA. However, most participants use DA incorrectly, which indicates a misunderstanding of the concept of perimeter as well as the concept of area, but also the dominance of PA. Observing the congruence of figure elements, moving and rotating them to compare figures mean using OA. However, operating with the figure did not lead to a successful outcome due to the wrongly used DA.

Final discussion

Based on the analysis of the presented results, it is evident that the participants were unsuccessful in solving the selected geometric problems, and the causes of poor success are many. They were the most successful in solving Task 1 and the weakest in solving Task 3, while they were almost equally poorly successful in solving Task 2 and Task 4.

Namely, in Task 1, only one geometrical concept had to be recognized - a triangle, which is used most often and continuously, and similar tasks are also used in mathematics classes. However, those who did not use the appropriate strategy and those with weaker visual-spatial skills were not successful in counting (Idris, 1998). In Task 3, poor performance may also result from weak visual-spatial skills, but also a weak ability to examine all possibilities, particularly because it was necessary to form different types of triangles and quadrilaterals.

In Tasks 2 and 4, the cause of poor success is primarily due to a lack of understanding of the basic concepts of perimeter and area of the corresponding figure and confusion about these concepts within one task (Tan Sisman & Aksu, 2015). Notably, in Task 2, the big problem is ununderstanding the concept of altitude, especially for the obtuse triangle (Miliković & Ševa, 2021). Furthermore, participants' poor performance is the result of the dominance of using formulas but also of calculating numerical values without prior visual processing of GF (Antunović-Piton & Baranović, 2022) and without a clear problem-solving plan (Baranović & Antunović-Piton, 2019, 2021).

To solve all four tasks, participants used geometrical figure apprehension according to Duval, but not in the appropriate way and not mutually. Firstly, because of their lack of conceptual understanding and secondly, because of perceptual dominance. Indeed, as Duval stated (2006): "Good conceptual understanding must lead the eye to what it needs to see to find out the elements necessary for the solution." Namely, insufficient coordination of discursive processing and operation with given figures (there is no mutuality between DA and OA) causes poor performance, which points to the problem of the dual nature of geometric figures (Duval, 2017; Fischbein & Nachlieli, 1998).

In particular, based on all the results in Task 4, it is evident that the offered answers somehow motivated the participants to answer the question (in item A, only one NO). According to their explanations, it has been observed that the answers partly direct their problem-solving process, but also, they try harder to explain (in item B, seven NO). Therefore, it is certainly helpful to look for explanations in the tasks with regards to offered answers, in order to gain insight into the participants' thinking and discover possible difficulties that hinder or prevent the problem-solving success.

CONCLUSION

The obtained results allow us to conclude that the selected sample of 75 Croatian pre-service teachers uses all types of geometrical figure apprehensions according to Duval's framework. Still, the type and the level of apprehension they use when solving a geometrical task have an important effect on their problem-solving process. In their reasoning process, perceptual apprehension is dominant and partially connected with others' GFA. This study follows previous research findings on GFA (Michael et al., 2011; Michael-Chrysanthou & Gagatsis, 2013). Their main obstacles in recognizing a geometric figure mathematically are: insufficient visualization skills; less ability to look at the figure in different ways; the constant need for numerical inference; visual-based estimating; inconsistency between verbal statements and the figure, the dominance of formula use and procedural computation without conceptual coherence with geometric figures in the problem-solving process; confusing concept of area and perimeter; inappropriate descriptive language and symbolic writing skills, in general.

Furthermore, students' perceptual apprehension runs against the mathematical way of looking at figures (Duval, 1995; Panaura & Gagastis, 2010), and the transition between different types of apprehension can help students understand how their geometrical reasoning is shaped (Duval, 2017).

In conclusion, one can develop and expand visual abilities and, consequently, geometric thinking by synergizing operational and discursive, sequential with perceptual apprehension constantly through the geometry teaching-learning process. Moreover, understanding geometric figures has a stronger relationship with operational apprehension but is often obscured by discursive and perceptual apprehensions (Michael et al., 2011).

Therefore, there is a necessity for teaching practices to include more activities that explore geometric figures, with a focus on mastering and developing the student's cognitive apprehensions for flexible use of different apprehensions, which will enable students to mobilize the proper way of looking at figures. One way is working with manipulatives and other teaching aids through all educational levels (Baranović & Lehman, 2017; Wanner, 2019). These activities can lead students to: acquire visual skills; enrich their language, which allows them to participate in discussions; introduce formal language naturally; develop a deeper understanding of concepts, rules and formulas; develop a problem-solving strategy; and stimulate creativity (Gridos et al., 2022).

Additionally, teachers need the knowledge and skills for such work, so pre-service teachers should also be taught this way in order to prepare themselves for future effective and comprehensive teaching. It would be recommended, for educational research, as stated by Duval: "The unusual way of seeing and complexity of the coordination between visualization and language must be a top priority for research into understanding and learning geometry processes." (Duval, 2014a, p.27).

It would be useful to research the extent to which in-service teachers know and implement visualization processes in working with geometric figures for the purpose of solving geometric problems and to gain a deeper insight into the way geometry is taught at all levels of education. In accordance with these results, guidelines can also be provided for their additional professional development.

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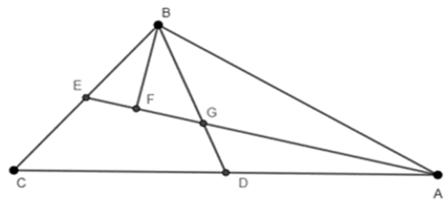
Appendix

Task 1. Counting triangles

a) How many triangles are shown in the figure?

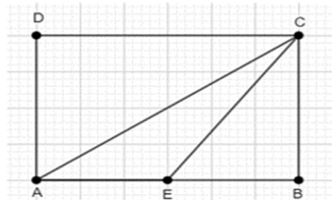
b) List these triangles.



Task 2. Area of triangle

The area of the rectangle ABCD is 24 cm^2.

Determine the area ΔAEC, if point E is the midpoint of segment $\overline{AB}$.

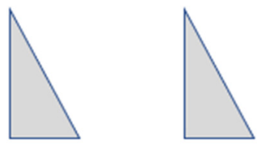


Task 3. Creating new figures

a) How many different figures can be composed of two congruent scalene right-angled triangles, provided that the triangles join along sides of equal length?

Answer: ___________________

b) Draw and name these figures.

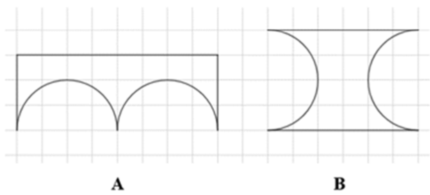


Task 4. Perimeter of figure

For figures A and B in the picture specify:

a) Which of the following statements is correct according to the perimeter of figures A and B?

1. Figure A has a larger perimeter than Figure B.
2. Figure A has a smaller perimeter than Figure B.
3. Figure A has the same perimeter as Figure B.
4. It cannot be determined because no measures are given.



b) Explain your answer.

TRANSFORMATION AS A STRATEGY IN PROBLEM-SOLVING

Jasmina Milinković

University of Belgrade, Teacher Education Faculty, Serbia

The paper outlines a strategy for problem-solving based on transformation. It provides a theoretical basis and practical tools for developing students' capacity to solve problems via transformation. Transforming a problem into a new one means that some of the elements of the problem space are changed while the others remain the same. Different kinds of transformation of problems in the process of problem solving are discussed. The first type is transforming a problem by changing the context, based on recognition of a mathematical idea in different contextual situations or different math domains. The second type of transformation is a creation of a new representation of the problem (symbolic, textual, visual). The argumentation was supported with exemplary cases of problem solving.

INTRODUCTION

Posing problems and solving problems are constantly and overwhelmingly present in mathematics education. "Students typically see mathematics as a body of knowledge used to solve well-defined problems uniquely" (Problem posing refers to the formulation of new problems and the reformulation of given ones (Chapman, 2012; Silver, 2015). It involves higher-order thinking skills and is recognized means of developing mathematical thinking and creativity in students of all ages (Koichu & Kontorovich, 2013). Often, mathematical problems represent special cases of general statements and truths that are the result of mathematical modelling (Ernest, 1991). Some mathematical models are the result of attempts to find resolutions and overcome the challenges of real life with mathematics.

The ability to solve problems using mathematical tools is one of the key objectives of mathematics education on all levels (MNPT, 2019, 2020). That is why so much attention is devoted to finding ways to improve problem solving skills of students. Problem-solving is the process of determining a sequence of actions accomplished to find a solution. To successfully solve classes of problems, it is necessary to build certain methods and strategies. Some problem solving methods are elementary school teachers' training: looking for a pattern, guessing and checking,

In I. Papadopoulos & N. Patsiala (Eds.), *Proceedings of the 22nd Conference on Problem Solving in Mathematics Education - ProMath 2022* (pp. 53–65). Faculty of Education, Aristotle University of Thessaloniki.

working backward, using Venn's diagram..., or creating a focused diagram (Dejić and Egerić, 2007).

Problem-solving strategy in an educational setting is a proposed scheme of actions developed to provide a path to finding a solution to problems. Each problem solving strategy includes multiple steps to provide guidelines on how to solve a problem. Effective problem-solving requires the identification of the problem, the selection of the right process to approach it, and the following of a plan (Polya, 1945). Regardless of knowledge of general strategies for problem solving, students often have difficulties in finding solutions (Galbraith, et al., 2006). Gagatis and Elia (2004) assert that every problem can be solved by using various types of representations implying that there is a close relationship between problem solving and representations. Along the line, multiple studies support the claim about the importance of representations in mathematics learning, since the use of various representations can help students solve mathematical problems (Bal, 2014; Dreher et al., 2016; Earnest, 2015; Flores, et al., 2015; Hwang, et al, 2007; Lesh, et al., 1987; Nistal, et al., 2009). There are multiple strategies in problem-solving in general and some of them are relevant for problem-solving in mathematics: visualization of the problem, drawing a diagram, breaking the problem into smaller pieces, working backward, and trial and error (Dejić and Egerić, 1992).

PROBLEM TRANSFORMATION

Problem space defines the problem. It is a description of given elements, unknown elements, and their relations as well as specifics of the context in which these elements exist. Any problem can be described in terms of its context, of givens and unknown elements, and the relationships between the elements. Context presents a set of determinants that define the problem situation, including the goal, constraints, and environmental conditions. From the problem space, we can discern the questions that may be asked. Sometimes, we are in search to find what is an unknown element, on other occasions we are trying to determine a relation between known elements.

Silver (2015) points out that posing a problem can occur before, during, or after the solution of a problem. He particularly pointed to re-formulation, as a form of problem posing occurring within the process of problem solving. He explains that when solving a nontrivial problem, recreating a given problem in some ways can make the problem more accessible for the solution.

Problem representation

Representations are considered a tool in thinking (Arcavi, 2003, Bruner, 1960, Couco & Curcio, 2001, Milinković, 2015, Michalewicz, & Fogel, 2004). A problem may be formed using different representations: symbolic, math contextual, realistic, and visually presented. Symbolic representation underlies the immediate application of learned procedures, mainly at the level of reproduction. Essentially, transforming a problem posed symbolically or textually into another form is a strategy of problem-solving based on changing representation. For example, to solve Problem 1, a problem solver needs to know how to find a missing element in a proportion. Thus knowledge of a mathematical procedure leads to successful problem-solving. There is no need for transformation.

Problem 1

$\frac{x}{2} = \frac{18}{6}$ x = ?

Solution

$$x = \frac{18 \cdot 2}{6}$$
$$x = \frac{36}{6}$$
$$x = 6$$

A problem set in a mathematical context underlies understanding the mathematical language and mathematical situations in which some rules are applied. They are frequently used in workbooks as well. These types of problems are used to practice the use of mathematical terminology as well as skills for applying procedures. For example, see Problem 2.

Problem 2 There are two similar triangles. Find the length of a side marked with "?". (Note that there is no need to measure the length of sides in drawings, the measurement is given and the drawing is a "visualization of the problem situation" which would otherwise be described in words. (Figures 1 and 2).

Solution

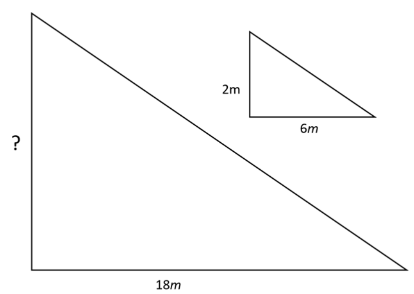


Figure 1. Visual representation of the problem with two similar triangles.

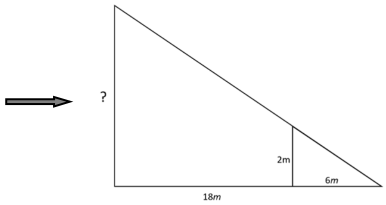


Figure 2. Transformed picture of the problem situation

$$\frac{x}{2} = \frac{18}{6}$$

To solve this problem, a student needs to realize that in the case of similar triangles, the ratio of matching sides equals the ratio of the other pairs of matching sides. Then, the problem is solved by creating a proportion and finding the missing element in it, like in the case described in problem 1. Thus, we transformed Problem 2 into Problem 1.

A visually presented problem incorporates an image, a graph, or other forms of visually presented information. Another example of a visually presented problem in a realistic context is one that we already introduced in different formats presented above.

Problem 3 Find the height of the tree based on the picture (Figure 3)

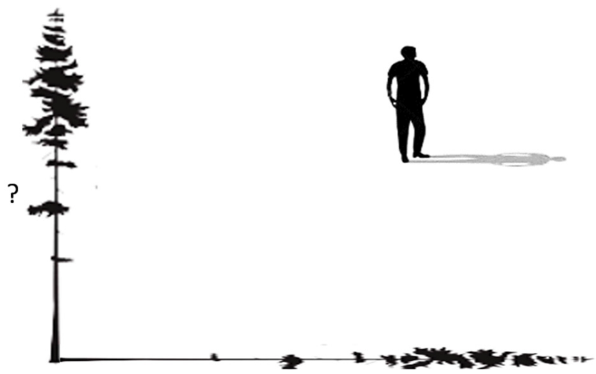
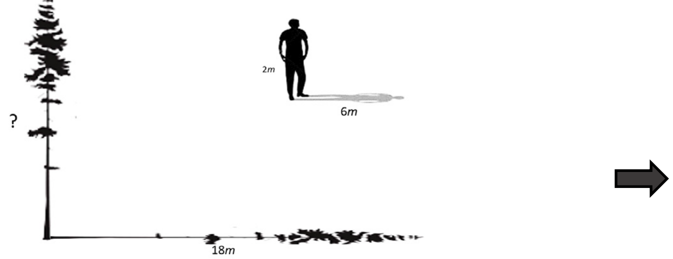


Figure 3. Problem situation in a realistic context with a tree and a man.

Solution



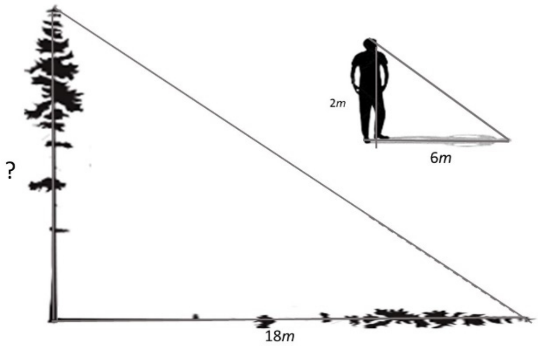


Figure 4. Transformation of the visual representation of the realistic context into a mathematical context.

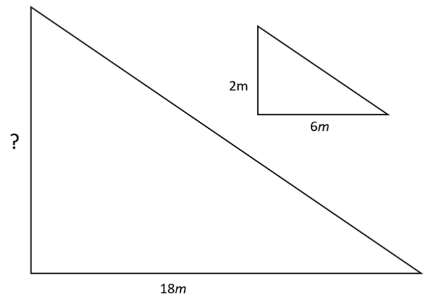


Figure 5. A transformed representation of the problem in a mathematical context.

$$\frac{x}{2} = \frac{18}{6}$$

The problem-solving process begins with an analysis of the situation presented in the picture with a tree and a man and their shadows (Figure 3). Measuring the given elements is the first step. The second step is to recognize that these objects and their shadows form a pair of similar triangles (Figure 4). Then, the transformed problem is solved using proportion (Problem 2) (Figure 5). Here, the problem with realistic context is directly related to the mathematical concept of the similarity of triangles. We assume that the solver would realize that the third side should be added to the drawing to form a triangle. This realistic problem can be easily solved by transformation into Problem 2, and then that one into Problem 1. A realistic problem requires applying knowledge and skills in a realistic context. Quite often but not necessarily, the plan for solving a realistic problem includes creating a graphic representation of the situation, like in Problem 3. In general, it is often not so obvious which math content is related to a particular problem situation.

Problem 4 A man 2 m tall is standing next to a pine tree. How tall is the pine tree if it casts a shadow of 18 m and the man casts a shadow of 3 m .

Problem 4 is a realistic problem in a textual format. It is a problem which analog to Problem 3, only the context is changed. The process of problem-solving would most probably start with drawing a visual representation of the problem situation. Then the process of problem-solving follows the steps described in the solution to Problem 3. Regardless of the specifics of a particular realistic problem, the problem solution most likely would proceed with a step of transforming the problem into a matching problem in a mathematical context. As we have seen in this sequence of various formats of the problem, a real problem is solved by transformation into a "math situation", visually presented, and then solved by transformation into a symbolically represented problem. Transformation into a visual form is not necessary but is often helpful in the search for connections to mathematical ideas. An example of transformations of a problem by varying context and representation is presented in Table 1.

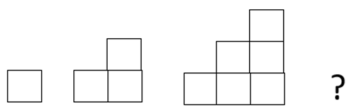
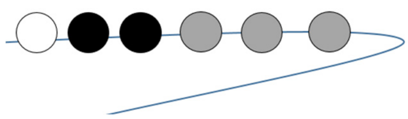
Math problem	**Context/Representation**
$\sum_{n=1}^{4} n$	Mathematical context, symbolic
Find the sum of the first four natural numbers.	Mathematical context, textual
	Mathematical context, visual
Ana creates a pattern of beads. She uses four colours. She used one white bead, two black, three red, and four blues. How many beads did she use?	Realistic context, textual
Find the pattern Ana uses to create neckless. How many beads in total will she use if she adds beads in another colour? 	Realistic context, visual

Table 1. Variations of a math problem based on context and representation

Ways of transforming problems in the process of problem solving can be related to the strategy for posing problems based on transformation. The reason behind the idea is that it is useful to consider transforming the problem by changing representation or some other elements of problem space so that it

- reflects the (changing) structure of mathematical knowledge,
- corresponds to different levels of students' knowledge,
- relates to different cognitive domains (knowledge, applications, reasoning).

There are several studies focused on different representations of problems(Niemi, 1996, Ikodinovic et al., 2019, Popović et al., 2022). In Niemi's study (1996), students were asked to represent their conceptual knowledge in several different task contexts and formats, and performance was compared across tasks. The results show that the level of representational knowledge predicts performance on problem-solving, justification, and explanation tasks (Niemi, 1996). Ikodinovic and colleagues investigated whether the representational context of a math problem affects students' problem solving (Ikodinovic, et al., 2019). They asked eight grade students to solve a set of problems. Each of the selected problems was presented in four variations of formats. For example, they posed a problem from the domain of equations in a symbolic format, textual with mathematical context, textual with realistic context, and visual with a mathematical context. They found evidence that the representational context of the mathematical problem influences students' achievement. The authors observed that students had particular difficulty with visually presented problems, which contradicts expectations based on earlier studies. Ikodinovic and colleagues (2019) explain that it might be a result of a lack of experience with the particular form of the problem. It goes along the line of our observation that such problems are rarely found in school textbooks. Popović and associates (2022) investigated students' transition from one representation of mathematical concepts in those problem formulations to another representation. They explored the influence of the representations used in the problem formulation (problems with the same mathematical background with regards to solving easier or more complex equations and determining the unknown value of the proportion) on students' success in solving those problems. On a representative sample of 584 eight grade students, they tested whether there were differences in students' success in solving mathematical problems while using symbolic, graphic, or verbal representations in the formulations of problems belonging to a different level of complexity. Results of this research indicate that there was a significant impact of the representations of mathematical concepts used in problem formulation on students' success. Heinze and colleagues (2009) found that problems in the form of symbolic representations are easily solved by students whereas, on the other hand, they have difficulties using verbal or graphic representations. The level of impact of using different representations in problem formulations depends on the level of the problem complexity when it comes to students' success in solving those problems (Heinze, et al. 2009). The studies show evidence that symbolically represented problems are solved more successfully. Visually represented problems and realistic problems often required multiple steps of transformation.

Thinking by analogy as a core of transformation strategy

Problem solving strategy of transformation is based on analogical reasoning. An analogy is a *comparison* between two objects, or systems of objects, that highlights respects in which they are thought to be congruent. *Analogical reasoning* is any type of thinking that relies on an analogy. An *analog*ous *argument* is an explicit representation of analogical reasoning that cites accepted congruities between two systems to support the conclusion that some further congruity exists (Paul, 2019). Problem-solving based on analogy is based on a comparison between two problem items, often the new and the old one for which we already know how to solve it. Thus, analogical reasoning helps the solver to find and generate a plan for solving the problem. You can use an analogy to simplify the problem you are trying to solve. To do this, the solver needs to compare the problem situation to a situation familiar to the past.

Transforming a problem in the process of problem solving by changing context is a kind of transformation strategy based on the ability of students to recognize the same mathematical idea in different contextual situations or different domains. Flexibility in thinking about the problem can result in a changed representation which eases the way to finding the solution. For example, finding the sum of the first *n* odd numbers may be solved using visualization. Finding the sum becomes equivalent to counting the number of knots in a square grid as it is done in Problem 5a).

Problem 5 *a)* Find the sum of the first n odd numbers.

Solution

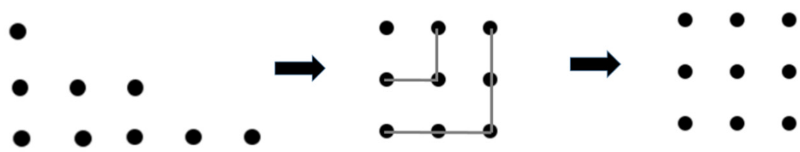


Figure 6. A process of transformation of the visual representation of the problem.

➡ $$1+3+5=3\cdot 3=9$$

In the general case,

$$1+3+5+\cdots+(2n-1)=n^2$$

To find the sum, we used an iconic form of the problem, representing numbers first as a sequence of points (Figure 6). Then comes a step of reordering points into a format of a square grid. This form leads to the conclusion that the sum of the first n odd numbers equals n^2.

b) Find the sum of the first n natural numbers.

Solution

Again, to find the sum of the first n natural numbers, it would be useful to transform the problem into a problem with a geometrical context.

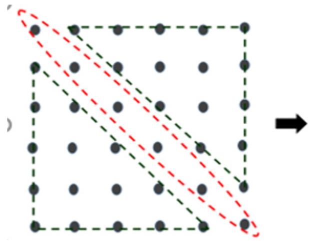


Figure 7. A visual representation of the problem

$$1+2+3+4+5=\frac{(5+1)\cdot(5+1)-(5+1)}{2}=\frac{(5+1)\cdot 5}{2}$$

The sum of n natural numbers is found using a square grid with $(n+1)\times(n+1)$knots (Figure 7).

Let me mention that going back to Ancient Greek, the Pythagorean school dealt in a congruent way with Problems 5a) and 5b) using visual representations to solve those problems. In contemporary times in Serbia, this type of problem is solved using mathematical induction in high school mathematics.

Transformation can result in a change of domain of mathematics or moving to a different domain outside of mathematics. Consider the following two problems and their domains. The solution for both problems is the same.

Problem 6 Determine the number of handshakes (Figure 8).

Figure 8. A combinatorial problem, Counting handshakes

Problem 7 How many straight lines connecting every two points can be drawn?

Solution

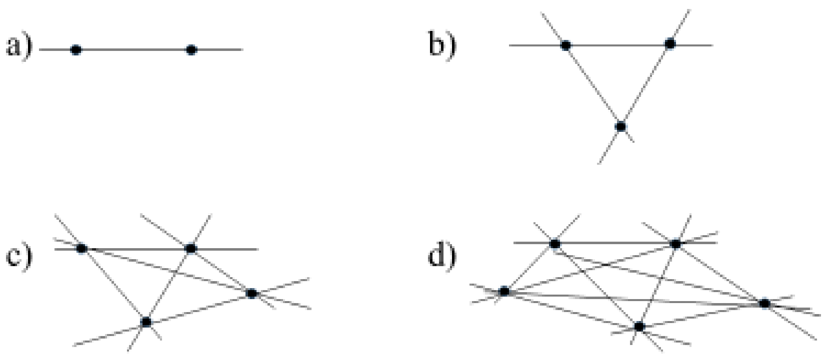


Figure 9. A combinatorial problem, Counting segments

One way to solve Problem 7 is to simply draw segments, hoping that all cases are accounted. This procedure would get too complicated with an increasing number of points. The same can be said for Problem 6 (Figures 8 and 9).

The other way to solve those problems is to recognize that in both cases it is a case combinatorial problem, a unique mathematical concept represented in different domains. The foundation of these problems is how to count the number of two-element subsets of a set of n elements. In Combinatorics, it is named *second-class combination of n elements*, n>1. In the case of a two-element subset, the number is

$$C_n^2 = \frac{n\cdot(n-1)}{2}$$

In the general case, the number of k-element subsets of the set of n elements is

$$C_n^k = \frac{n\cdot(n-1)\cdot\ \dots\ \cdot(n-k+1)}{k!}$$

In the case of segments, from each of the n given points, it is possible to draw n(n−1) segments. But in this way, each segment is counted twice (AB and BA are the same segments.) That's why it is necessary to divide the total by 2. The solution for the problem with handshakes is analogous. This solution for two problems set in different contexts is an example of mathematical modeling.

The curriculum for primary grades in Serbia explicitly states two problem solving strategies for solving algebraic problems using two methods fitting to the transformation strategy, the *Segment method* and *the Rectangle method* (MNPT, 2019, 2020). Both methods are based on the idea that some types of algebraic problems can be solved by finding appropriate geometrical representations. Then, finding a solution using knowledge of geometry assures finding the solution for the original algebraic problem. The *segment method* is often used in situations when quantities can be represented by segments accordingly (e.g. equal quantities figuring in the problem space are represented with segments of equal length.) The *rectangle method* is used when data is in a relationship that can be represented as a multiplicative scheme. The relationship between elements in that case can be represented with a rectangle or a square. These two methods of problem solving are recommended to be introduced in primary grades by the Serbian mathematics curriculum for primary grades (MNPT, 2019, 2020).

IMPLICATIONS AND CONCLUSIONS

Throughout the paper, we pointed to the possibility of linking strategy in problem posing with a strategy in problem solving. We were particularly focused on transformations of the problem as a strategy in problem solving. The argumentation using exemplary problem situations and description of problem solving process via transformation proves that the transformation strategy can be useful. In Serbia, to some degree, it is introduced in the curriculum and thought in primary grades (MNPT, 2019, 2020). Solving problems presented in various formats, with realistic or mathematical contexts helps students to expand their mathematical modeling competencies. That is why, I believe that problems should be given in different formats and representations including textual form, pictures, tables, or graphics.

The potential of teaching transformation strategy should be empirically investigated in a classroom and compared to other strategies of problem solving. The transformations strategy could help students make connections and advance problem-solving competencies. Problem solving remains a creative endeavour even with well-thought strategies.

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HOW MATHEMATICS TEACHERS UTILIZE LEARNING OUTCOMES ON ALGEBRAIC PROBLEMS IN THEIR TEACHING SCENARIOS IN A REFORM-DIRECTED CONTEXT

Christos Souralis and Chrissavgi Triantafillou

National and Kapodistrian University of Athens, Greece

This paper explores how teachers use specific learning outcomes on algebraic problems in their teaching scenarios in a reform-directed context. The context of the study was a professional development program aiming at familiarizing a group of in-service mathematics teachers with the mathematics curricula recently introduced in the Greek secondary education but without the support of a textbook. The research data was participants' four teaching scenarios. The results indicate that participants developed ways to support students' explorations and modelling of daily life situations but in diverse ways. Main task designs differ on their inquiry characteristics. Their planned teaching practices differ on their suggested use of tools and on the coherence of the context of the main task and the assessment tasks.

INTRODUCTION

The teaching of algebra incorporates the emphasis on patterns and generalization as well as mathematical modeling of real-life contexts (e.g., Fey & Smith, 2017). Changing algebra teaching with emphasis on teachers' problem-solving instruction forms a widespread research field (e.g., Lester, 2013) and is in alignment with the changes in the teaching of algebra proposed in the recent Greek reform-directed curriculum.

This study focuses on how a group of in-service mathematics teachers use the reform-directed curriculum materials in their teaching scenarios during a Professional Development (PD) program. Since no textbooks are written yet, participants had to develop their teaching scenarios exclusively on specific Expected Learning Outcomes (ELOs). ELOs are reformed oriented statements that express what students are expected to know, to do, and to understand upon completion of a learning experience in a measurable way (Divjak & Ostroški, 2009; Kennedy, Hyland & Ryan, 2009). Thus, ELOs are related to teachers' task designs and classroom practices including their evaluation techniques. While the concept of outcome-based education is high on today's education agenda (Kennedy, et al., 2009) it is an under researched area on how teachers enact the

In I. Papadopoulos & N. Patsiala (Eds.), *Proceedings of the 22nd Conference on Problem Solving in Mathematics Education - ProMath 2022* (pp. 67–78). Faculty of Education, Aristotle University of Thessaloniki.

specific curriculum resources in their classroom teaching. In this study we focus on ELOs related to problem solving and modelling practices.

Specifically, our research explores how in-service mathematics teachers in a reform-directed context utilize ELOs related to algebraic problems in their teaching scenarios. In teachers' lesson designs someone can identify their decisions about various aspects of instruction in the context of a reform-directed context (Superfine, 2009).

The emergence and development of algebraic thinking in a problem-solving context are connected firstly to the nature of the tasks that are given to students and secondly, to the repertoire of teachers' instructional techniques (Bednarz & Janvier, 1996). Thus, this study focuses on the characteristics of the main tasks that teachers based their teaching scenarios and aspects of their teaching and assessment practices.

LITERATURE REVIEW

On problem solving and modelling

Pólya (1945) describes the problem as an original mathematical task where the students do not know how to solve it. Many types of problems are described in the literature. An open-ended problem is a problem that has many solutions, the solver can deploy multiple approaches to the solution, or the solver can change the task's parameters (Chan & Clarke, 2017). Modelling problems are problems framed by a context that can be a problematic situation of the daily life which calls for the design of a mathematical model to describe and resolve it (Anhalt & Cortez, 2016; Czoher, 2018; Hsu et al., 2007) or an authentic professional practice that calls for the design of a model as well (Vroustis, Psycharis & Triantafillou, 2022). A modelling problem engages the solvers in a process that transforms a real-life situation into a mathematical problem (Kaiser & Schwarz, 2006). Authenticity is a term Palm (2008) uses to describe the gap between reality and a text-based problem and is framed by the following 5 dimensions: The event described in the task; the purpose of the task that describes the appropriateness of answering the question in relation with the context; the existence of the question in the reality; the language/terminology used and the authenticity of the given information/data the task is based on.

In a modelling lesson the teacher is called to make informed teaching decisions on the spot and such teaching practice can be different than in a traditional mathematics classroom since students' creative activities are unpredictable since students can present multiple solutions and strategies (Alwast & Vorhölter, 2022). The literature proposes different tools to be used by the students during their engagement with the modelling problem such as digital tools like geogebra, casyope, and manipulatives (Kafetzopoulos & Psycharis, 2022). Also, the literature proposes as a main mathematical concept in algebraic modelling problems the mathematical concept of covariation reasoning (González, 2021).

On enacting reform -oriented curriculum resources

Learning can be studied from the design point of view and is supported by resources of various genre like commercial supportive texts or curriculum resources (Remillard, 2012). Curriculum resources are widely regarded as instruments for the implementation of change in mathematics classrooms (Cai & Howson, 2013; Remillard, 2005). Curriculum resources are considered as educative for teachers where the adjunct 'educative' refers to teachers as learners since these materials support teachers' knowledge and professional development (Rezat, Fan & Pepin, 2021). The relation between teachers and curriculum resources can be viewed as a participatory relation where the curriculum materials are artefacts that mediate reform and their purposeful use by teachers/designers can shape their practice.

Review research on the relation between teachers and the curricular resources is reported by Remillard (2005) who identifies the following four dimensions: *fidelity* between the curriculum materials and teachers' planning to transfer the curriculum ideas to their students; *teachers' decision* on which of the available resources to draw on while creating their teaching designs; their *interpretation of the specific curricular* resource by taking into considerations their views and believes on teaching, on mathematics, and on curriculum usage; the *enacted aspect* of the curriculum. Most studies explore the relation between teachers and textbooks (e.g., Rezat et al., 2021; Chowdhuri, 2022). Particularly, Rezat et al. (2021) explore the agency of the text and the influencing factors in teaching practices. In times of curricular change, curriculum resources are influential, but they alone cannot change teaching nor learning practices while more research is needed about the features of curriculum resources that support the implementation of change (ibid). Chowdhuri (2022) investigates how textbooks in a curriculum reform in India inspired by critical perspectives represent and communicate the above aim to their readers. The results revealed that textbook use a radically unique 'voice' (Remillard, 2005) to introduce school mathematics with authentic and socially relevant contexts within their tasks.

The current study explores how teachers use curriculum in the form of specific ELOs, as their main curricular resources in their teaching scenarios without the use of a textbook to support their teaching designs.

METHODOLOGY

The context of the study

The study took place during a PD program in a group of in-service mathematics teachers who are teaching in Greek experimental low secondary school. The aim of this session was to inform teachers on aspects of the Greek reform curriculum.

One of the authors was the Teacher Educator in this group. The PD session lasted for 7 weeks, 2 hours per week.

The main reform-directed resources used in the PD session were the teacher guide, the specific ELOs introduced in the reform curriculum, and text materials on the general aims of the new curriculum, on assessment and on differentiation and inclusion. In the teacher guide specific tasks are presented and their relationship with ELOs. We must add that in the time of this study no textbook was available for teachers. In the end of the PD program participants had to develop a teaching scenario. Two or three participants could develop the same teaching scenario. The teaching scenario is a document that includes information about the school context, the main mathematical tasks and the ELOs they refer to as well as the assessment tasks and the proposed teaching practices. Every teaching scenario should fulfil certain ELOs relevant to the new curriculum. The teaching scenarios were the final written task the mathematics teachers delivered during their participation in the PD program. Participants were free to choose specific ELOs on a mathematical topic to develop their teaching scenario.

Participants

The participants were 19 practicing mathematics teachers in Greek experimental public schools. They worked in groups and developed 9 teaching scenarios. Six in the field of Algebra and three in the fields of Geometry or Stochastic mathematics. Four of these scenarios are based on ELOs related to problem solving and/or modelling on specific thematic units.

Research Data and Data analysis

The research data were the teachers' teaching scenarios. The following steps of analysis were made. In the 1st step, we identified within every teaching scenario the ELOs participants based on their scenario. Then, we distinguished the ELOs related to problem solving and/or modelling. These ELOs were the following: Al.7.3. *To solve realistic and mathematical problems using arithmetic and algebraic expressions*; Al.8.1. *To recognize the covariation of quantities in everyday situations and distinguish which quantity determines the other*. Al.8.7. *To recognize in different contexts the proportional relation between quantities*.

In the 2nd step, we analyzed the main tasks' features, teachers based on their teaching scenarios. We used the following dimensions to characterize each main task: the *mathematical topic*, the *context* which is the problematic situation the students are engaged in, the *type* of the problem (closed or open-ended), the *data* which refers to the type of the given information (authentic or simplified), the *use of representations* which refers to the role representations have (vital or supplementary), the requirement for students to develop the *modelling practice* or not. In the 3rd step, we analyzed how the teachers incorporated algebraic problems in their teaching. We described the type of *tools* proposed to be used by the students to solve the problem (e.g., in a static of dynamic environment), the *classroom organization* (individual or teamwork), and how the specific ELOs are realized in students' assessment practices.

RESULTS

Teachers could choose a task from a given pool of tasks or they could design a task themselves. The following tasks were either self-generated or were based on other resources besides the ones provided to participants in the PD program or the school textbooks they were currently using.

Teachers developed the following main tasks in their teaching scenarios.

Georgia's bike motion. Students are given a description of Georgia's bike motion that includes distances, stops and times. They are asked to use a GeoGebra applet to model the situation by designing a graph that describes Georgia's bike motion (Figure 1).

Georgia started cycling at a steady pace from her home to the Super Market, which was 1.2 km away but halfway through, after 3 mins she saw her friend Maria and stood to chat with her for 4 mins. Then she continued to the supermarket at a steady speed. The ride, from her home to the Super Market took her 13 mins total.

To describe using the app https://www.geogebra.org/m/dncfgep5 to graph Maria's bike ride.
(You can move points B, C, D and see what happens by pressing Start).

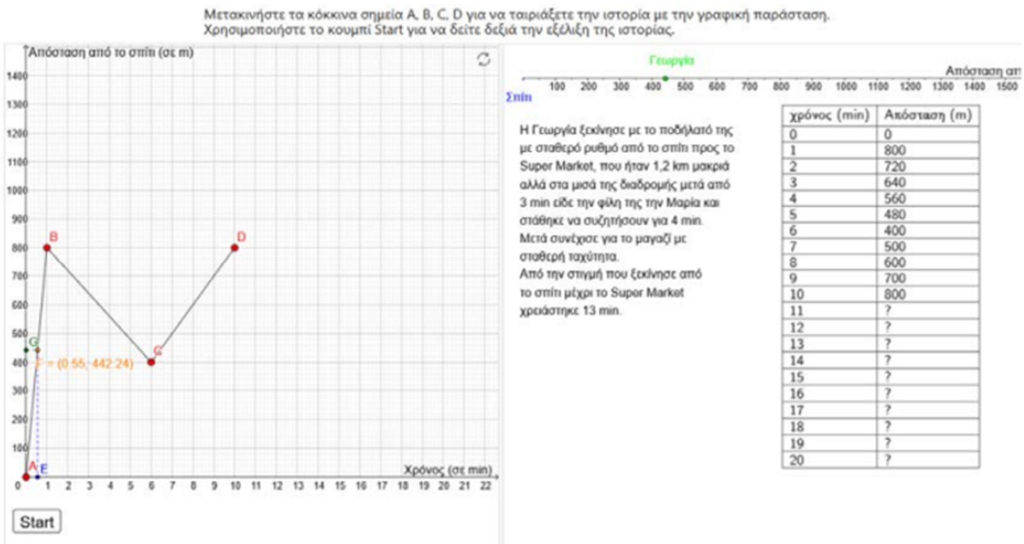


Figure 1. Georgia's bike motion

The calendar. Students are given a screenshot of a windows calendar and they are called to explain how one can know which four dates are creating a given sum under the assumption that the four dates are forming a square on the calendar. Students are called to model the situation by recognising numerical patterns on the calendar's dates and solve linear equations (Figure 2).

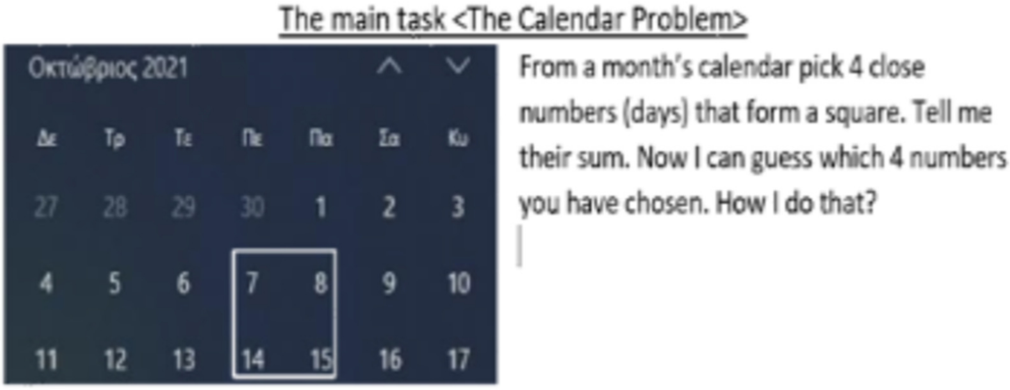
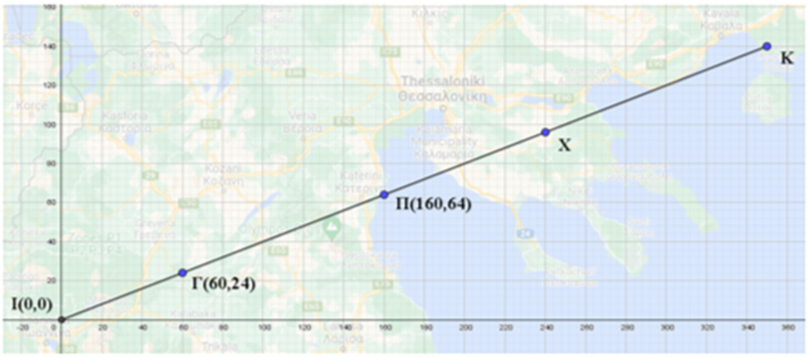


Figure 2. The calendar problem

Task A. An airplane operates the route Ioannina (King Pyrros airport) – Kavala (Alexander the Great airport) on a straight-line airstrip as shown on the map below. The control tower of Ioannina airport records the coordinates (x,y) of the plane (in kilometers) throughout the route Ioannina (I), Grevena (G), Pieria (P), Chalkidiki (C), Kavala (K).



a) Which of the functions represent the plane's path.

A. y=x B. y=x−36 C. y=x+20 D. y=0,4x

b) Fulfil the following matrix. What do you make of this?

Town	Γ	Π	X	K
X	60		240	
Y		64		140
$\frac{y}{x}$				

Figure 3: The airplane rout

The airplane route. The data provided to students are the graphical representation of an airplane route that passes from specific Greek towns and the cartesian coordinates of each town in reference to a geographical map. The coordinates show the distance between the towns in Kilometres. On the x-axis is the west/east distance between the towns and on the y-axis the north/south distance between the towns. All towns are placed on the same straight line. The students are called to model the situation by identifying the symbolic representation of the specific graph (Figure 3).

The farmer's rabbit farm. Students are given the possible dimensions for building a fence around a rabbits' farm, suitable for domestic production. The farmer also needs to build on the farm a rectangular or a square rabbit shelter, fenced by using a maximum of 36 meters of barb wire. The rabbit house should be made according to specific dimensions. The students are called to model the situation and find a way to calculate one side's length given the length of the other side of the fence. Additionally, a worksheet will be given to students that will guide them on calculating specific values for the length and the width of the fence (Figure 4).

A farmer wants to fence part of his estate with barbed wire to build a rectangular or square space to where he will breed rabbits. Inside that space, a rectangular area where he will build a shelter for his rabbits measuring 3.5 m long and 2.5m wide should be included. It has 36 meters of barbed wire and the necessary space to implement its construction.

A) He wants to find a method that will enable him to calculate one dimension of the fence given different values of the other dimension. That will help him to decide what the ideal dimensions will be. He turned to his child, who is in grade 8, for help.

B) How the value of the area of that fenced space is related to the values of its dimensions?

Figure 4. The farmer's rabbit farm

The related to problem solving and/or modelling ELOs, teacher utilized in their teaching scenarios were as follows: *Georgia Bike motion* and *Farmer's rabbit farm* was Al.8.1; *Airplane route* was Al.8.7; and *the calendar* was Al. 7.3.

The characteristics of the suggested tasks

The mathematical topics teachers chose for developing their teaching scenarios were *linear functions and linear co-variation between quantities*; *linear equations*; and *numerical patterns*.

Characteristic examples on linear functions were the 'Airplane route' and the 'Georgia's bike motion.' In these tasks, students will work with symbolic and graphical representations. The 'Calendar' synthesizes the notions of numerical patterns and linear equations. In this problem, students will find that dates forming a square always form a pattern. For example, if the first date is x, the next date could be x+1 and so on. If someone knows the sum of these dates, they can find the specific dates on the calendar by solving a linear equation.

The context of all the problems was *Daily life* situations.

Two of the algebraic problems were *open-ended*: The Calendar, where the students will potentially employ different solution strategies, from empirical to symbolic to respond to the task. The 'Georgia's bike motion' problem, where the solvers are asked to graphically describe a movement phenomenon without supplying any pre-determined steps.

The use of representations was either *vital*, for example, the 'Calendar' and the 'Georgia's bike motion' or *supplementary* (e.g., the 'Airplane route').

The data was *authentic* or *simplified*. The data were authentic only in the case of the 'Calendar' problem since the participants provided a photo of a specific month (October 2021). The data was simplified in all other cases. For example, in the 'Airplane route' problem it seems that the airplane is moving on a completely straight path between different cities.

Teachers' planning to incorporate the algebraic problems in their teaching practice

The proposed teaching tools to be used by the teacher or by the students are:

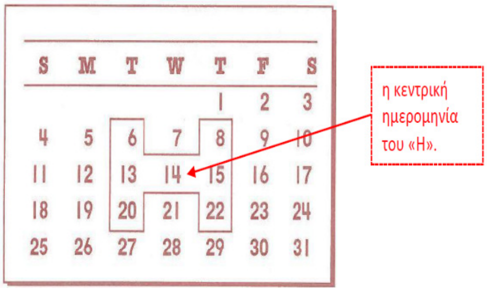
Digital: interactive board, calculators, GeoGebra, and smartphones used by students in the 'Georgia's bike motion', and the 'Airplane route'.

Non-digital: cards with supportive questions mended to be given to specific students who will need further support were used in the case of the 'Calendar' problem, and the use of grid paper was proposed in the 'Farmer´s rabbit farm'.

The classroom organization was *teamwork* usually combined with whole class discussion at the end of the classroom activity.

Teachers' proposed assessment practices

The *ways of assessing students' understanding* were either problems that had the same mathematical topic with the main task but introducing students in a different situation or tasks that are extensions of the main tasks. For example, the assessment task in the 'Farmer's rabbit farm' scenario was a problem asking students to find the error of the barcode machine that a sales controller should handle. Thus, students will be engaged in the same mathematical topic i.e., linear covariation between quantities but in a different daily life situation. In the case of the 'Calendar' problem, the assessment task was an extension of the main task where the mathematical topic and the context remained the same. In this task students will be asked to find other patterns on a calendar (Figure 5). In this case there is an alignment between the context of the main task and the assessment task.



Calculate the sum of the 7 dates. Then make an equation so that if someone solves the equation, they will be able to calculate the central date of H.

Figure 5. The assessment task of the 'Calendar' problem

CONCLUSION

This study explores how in-service mathematics teachers in a reform-directed context use the same or relevant learning outcomes related to algebraic problems and/modelling in their teaching scenarios. The mathematical topics teachers based their scenarios on were mostly on linear functions and numerical patterns. Teachers in their effort to address the specific learning outcomes developed a distinctive algebraic problem with a variety of characteristics. The fidelity dimension was fulfilled in all teaching scenarios since the context of all the problems was exclusively based on daily life situations and all the scenarios were designed to engage students in modelling practice. Thus, participants fulfilled the basic requirement of the specific learning outcomes as it is proposed in the relevant literature (Hodgson & Wilkie, 2022; Jung & Brady, 2016; Floro & Bostic, 2017).

On the other side, teachers' decisions on which algebraic problem to use in their teaching scenarios and the interpretations of the specific learning outcomes were varied as reported in Remillard (2005). Particularly, we see different main task characteristics, since for example only two of the problems were open-ended, the data were mostly simplified while the use of the representations was either vital or supplementary. The open-ended tasks, the authenticity of data and the vital role of a representation in a problem-solving situation characterize an inquiry activity (Calleja, 2019) that is also recommended by the new curriculum.

In the suggested task enactment, teamwork seems to be the preferred teaching practice by all participants, a result that follows the idea that making new connections between pieces of mathematical knowledge is triggered through social interaction (Zbiek et al., 2022) and is also in line with the reform curriculum. There was also a variation in the proposed teaching tools from digital to conventional ones. The assessment tasks teachers proposed in their scenarios mostly are designed to engage students in the same mathematical topic but in different everyday situations.

In general, participants used the same or similar learning outcomes in diverse ways in their teaching scenarios. Thus, we argue that the absence of a textbook and the need to design a lesson exclusively based on specific learning outcomes provided teachers with authentic opportunities to develop ways to support students' opportunities to explore and model real word situations but in diverse ways. Particularly, participants' main task designs differ on their inquiry characteristics (e.g., use of authentic data versus use of simplified data; use of open-ended versus closed problems). Their planned teaching practices differ on their suggested use of tools (digital versus traditional) and on the coherence of the context of the main task and the assessment tasks.

In the present research, we focused on the teachers' designs, but we haven't analyzed any data from the implementation of the teaching scenarios usage in class. It would be of great research interest to study the classroom interactions and the outcome from the usage of similar teaching designs as proposed in the literature (e.g., Vroutsis et al., 2022).

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GROUP–2

Investigating problem solving from the students' point of view

PROBLEM SOLVING WHEN DIGGING FOR FRACTIONS: A CASE STUDY OF A FRACTION NUMBER LINE GAME

Georgios Thoma[1], Julia Bahnmueller[1,2] and Korbinian Moeller[1,2,3]

[1]Centre for Mathematical Cognition, Loughborough University, United Kingdom

[2]LEAD Graduate School and Research Network, University of Tübingen, Germany

[3]Leibniz-Institut fuer Wissensmedien, Tübingen, Germany

Complementing existing evidence from quantitative studies evaluating the effectiveness of the fraction number line game Number Trace, the present paper provides a qualitative analysis of five participants, aged 9 to 15, during their first encounter with the game. Continuous screen-capture data allowed for a detailed description and analysis of players' emerging problem-solving methods and techniques from a local-global perspective. Overall, the present study helps in better understanding how games, with their particular set of affordances and intrinsically integrated characteristics, can be described as an efficient problem-solving environment to facilitate (fraction) learning.

INTRODUCTION

Problem-solving skills are regarded as one of the 21st century skills which are key to life prospects in the digital age (Van Laar et al., 2017). Contemporary research on the subject focuses among others, on the role of technology in problem solving, highlighting the affordance of constant and instant feedback, the exploratory aspect of digital technologies, as well as the iterative process in a constrained task (Liljedahl & Cai, 2021). Similarly, in their systematic review, Pan et al. (2022) showcase how video games engage players into knowledge and skill acquisition of mathematics. Thus, it may be fruitful to consider problem-solving within game-based learning and to investigate the potential of educational games as problem-solving environments.

In this qualitative study, we observed students while they were playing an educational game for the first time. With this study, we aimed at answering the following two research questions:

1. Can an educational game—which was repeatedly shown to foster children's fraction understanding (e.g., Ninaus et al., 2017; Kiili et al., 2018b)—be conceptualised as a problem-solving environment?
2. What affordances and opportunities may an educational game provide for problem solving?

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THEORETICAL BACKGROUND

Problem Solving

Polya (2004) differentiates between different steps of solving a problem: first, the need to understand the problem, then the creation of a plan using available data, followed by carrying out the plan, and finally looking back and reflecting. These steps illustrate the cyclical nature of problem solving. Downs and Mamona-Downs (2007) further suggest that a problem can be described from a local and a global perspective. Locally, an individual problem is specific and unique. Globally, a problem is also part of a system of more global rules and principles. Attempting to solve a problem locally may lead to the development of problem-solving approaches that may be generalized to a more global scale, potentially assisting to solve similar local or completely different (global) problems. The application of specific methods and techniques are two of these approaches (Mamona-Downs & Downs, 2004) that emerge locally yet may have a global impact. Both methods and techniques convey mathematical arguments and can be applied in diverse sets of tasks. While a method is abstract and unorganized, a technique can be described as sitting between a heuristic and an algorithm (Mamona-Downs & Papadopoulos, 2017). This means a technique is more structured and includes arguments that are divided into stages that address specific aims. Techniques can form during any problem-solving stage, as well as when a solution has been found. Problem-solvers are usually unaware that a technique has been formed unless they examine past experiences or solutions. Once identified, techniques can become tools to solve similar or different problems with common attributes. In line with the literature, we will consider participants' approaches as *methods* when these are unorganized in their initial stages and use the term *technique* either when *methods* are consistently combined as one, or when a *method* has evolved to a recuring procedure.

Game Elements

To better understand games as problem-solving environments it is important to consider the essential elements of a game (Plass et al., 2015). At the core of a game are the game mechanics. These determine how players interact with the game (e.g., moving a game avatar) and thus, influence gameplay. Moreover, visual and musical art is used to communicate different types of information of the game world to the player through the respective senses. To keep players engaged and motivated, different virtual incentives exist (e.g., in-game points or collectibles). Finally, the game's narrative, which serves as a cohesive story in which the gameplay is integrated. Depending on the availability of resources and game design, these five elements (game mechanics, visual, musical art, incentives, narrative) are combined at various levels of cohesion and artistry to make up a game. However, in the context of games created for educational purposes—in the present case, mathematics—an extra element exists: educational content. This content is the encompassing element that needs to be properly conveyed through all other game elements for the game to succeed in its main purpose of supporting the learning of said content. While studies have critiqued the use of game elements for educational purposes, suggesting that players might be distracted by them (e.g., Mayer, 2020), it has been argued that if game and learning mechanics are properly aligned (i.e., intrinsically integrated; Habgood & Ainsworth, 2011), this should provide successful learning.

Fractions

Shown to be significant predictors of future mathematical achievement (Siegler et al., 2012), fractions are a topic which students but also adults struggle with considerably (DeWolf & Vosniadou, 2015). This is due to a variety of misconceptions that arise when learners attempt to overcome the incompatibility of rational numbers in a whole number setting—for example when performing basic arithmetic operations with fractions. In formal education, fractions are commonly introduced through area models and part-whole relations (Fazio & Siegler, 2011). However, research suggests that the number line is a powerful tool to teach fractions, which amongst other advantages, introduces rational numbers by expanding the concept of the number line for natural numbers (Sidney et al., 2019). Thus, common mathematical tasks in schools are carried out on a number line—such as identifying fractions, comparing, ordering, estimating and locating fraction magnitudes (Fazio & Siegler, 2011).

Tasks like these have become an inspiration to research teams for creating (fraction) number line games. For instance, Kiili et al. (2018b) ran a training study on fourth graders (average age 10 years) in which the training group played the fraction number line game five times for half an hour each while the control group attended typical math lessons. Pre- and post-tests comparisons indicated a larger increase in fraction understanding for the training group (for magnitude comparison and ordering). Additionally, Ninaus et al. (2017) assessed participants fraction knowledge, reporting a significant corelation between in-game performance and math grade in fifth graders (average age 11 years), showcasing it as a valid assessment tool. Expanding on the above, the game was first used cooperatively in student classrooms (with 10 and 12-year-olds), and then competitively, examining participants engagement and intrinsic motivation (Kiili et al., 2018a).

Being indicative that the game can be effectively used as both a training as well as an assessment tool, in the current study we used the game's latest version *Number Trace*, to investigate the affordances and opportunities it offers from a problem-solving perspective.

METHODOLOGY

Participants

Participants taking part in the study were five boys between 9 and 15 years who were recruited as a convenience sample through personal communication. The participants—Luke, aged 9; Jeffrey, 11; Malcom and Max, 13; and Ben, 15—had never played the game prior to the study. Our aim was to investigate participants' approaches and comments, regardless of age group or prior experience with fractions. As such, no math test or grades were gathered from participants. However, a basic conceptual understanding of fractions was considered necessary, so participants would be able to play the game just by the instructions given. Therefore, we recruited participants from late primary and early secondary school. As this is the first qualitative study on this game, the resulting wide age and competency range is well suited to gather a diverse and rich set of problem-solving approaches.

Prior to data collection, parents and participants gave their written consent to participate in the study. The study was approved by the ethics subcommittee for studies involving human participants at Loughborough University (UK).

Setting of the study and data collection

The study took place online. Each participant was briefly interviewed in the beginning of the session. Participants were asked about their age, prior experience with games (in and out of school), and what they already knew about fractions. The aim of this interview was to warm-up participants, communicate the aim of the study, explain data anonymisation, introduce the task, and check for any technical issues. On average, the study took about 30 min, of which participants played the game for approximately 20 min (playing 7 levels of 12 fractions each).

Participants were instructed to articulate their thoughts and explain their actions as detailed as possible during gameplay. After the movement keys were explained, participants began playing the game while their screen was being shared with the researcher and their on-screen actions and audio were recorded. As the focus of the study was to evaluate the application of problem-solving strategies (and less so participants' performance in the game), participants were interrupted at times while playing to allow for short clarification questions on their actions or to prompt elaborations of their approach in key moments. After their playthrough was completed, a final short interview followed. This interview included questions about their overall experience, about favorite and least favorite moments or interactions, if there was anything they wanted to do that the game didn't let them, and what they would change, if they could change any aspect of the game or their experience.

Description of the game

Number Trace is a number line estimation game in which players estimate the position of fractions as accurately as possible on a number line typically ranging from 0 to 1 (video trailer: Kristian Kiili, 2018). Different aspects of *Number Trace* have been evaluated quantitatively in several studies (e.g., Ninaus et al., 2017; Kiili et al., 2018b). While the overall game design was consistent across studies, slight variations may have been implemented to address the respective research questions.

The number line estimation task is the core game mechanic integrated intrinsically in the narrative of the game which builds on the classical animal enmity between cats and dogs. An angry cat has hidden all the bones and a dog needs to find them by digging at the location that is hinted by a fraction. The more successful a player estimates the position of a fraction on the number line, the more bones are awarded. Thus, bones are used as a virtual incentive but also as a measure of game performance. Starting each trial, the dog is positioned at the 0 mark. To estimate the position of a fraction, the player manoeuvres the dog avatar across the number line using the "left" and "right" arrow keys on a standard keyboard. The spacebar is used to confirm the location where the dog should dig. To assist in precise digging, a yellow point nested in a short vertical white line signifies the dog's position as it moves across the number line (visible in Figure 1 on top of the 0 mark).
In case the estimation deviates from correct position of the fraction on the number line by more than 8%, the dog shows signs of sadness, and no bones are awarded (see Figure 2a). If the dog digs close to the correct position of the respective fraction (i.e., less than 8% from the correct position), it shows signs of happiness and bones are awarded (see Figure 2b,c,d).

Figure 1. The game. (a) The fraction estimation task. (b) After completing a level, stars are awarded (one star for completion, two for 3200 bones gathered, three stars for 5000 bones), and the narrative is visually reinforced.

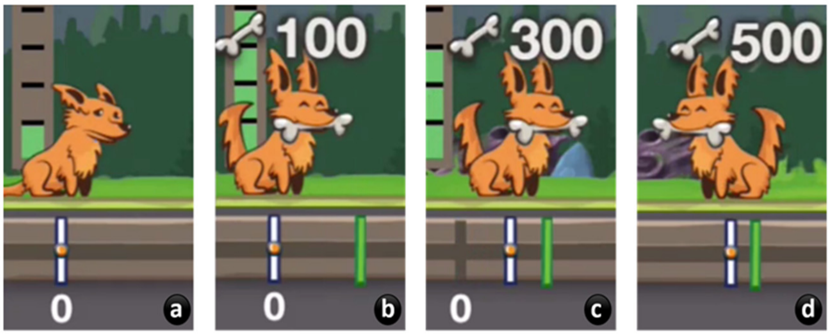

Figure 2. Feedback and reward. Depending on the estimation error, animations and bones awarded vary: (a) the estimation error is ≥ 8%, (b) < 8%, (c) < 5%, (d) < 2%.

ANALYSIS OF THE DATA

Gameplay recordings was transcribed, alongside the notes taken by the researcher during sessions. Recordings were then revisited highlighting points of interest regarding participants' comments and behaviour during gameplay. Special attention was paid to behaviour indicating the possible existence/development of a method or a technique or situations considered to be important from a mathematics education perspective (e.g., mentions of simplification, an equivalent fraction or a fraction used as a benchmark). Patterns between participants were then identified, points of interest were revaluated and compared. In the numbered transcripts below, [act] is used to highlight significant actions. For greater clarity towards understanding the transcripts and due to overlaps in the actions presented, in addition to the numbered transcripts, timings are offered.

For the purpose of this study, music or sound effects were turned off, allowing for better audio quality during the think-aloud phases.

Luke's first playthrough

Out of all participants, the youngest, 9-year-old Luke's cheery disposition and eagerness to discuss serves as an excellent representative example to showcase *Number Trace* as a problem-solving environment, both due to him being verbally expressive as well as due to the events exhibited in his first playthrough.

Luke (L) was highly talkative and a frequent unprompted commentator, showed great engagement with the game and kept comparing his scores from level to level. After giving the introductory interview and listening to the instructions by the Researcher (R), L begins his first level of *Number Trace* (NT).

0	NT:	(0:00) 1st fraction shown: 9/16, 0 total bones (see Figure 1a).
1	L:	(0:00) "I see like a dog, I think, and a chart with a flag. So I see "1", that is level one, and I start on "0" and yeah, the background that looks like a forest." (Luke moves his mouse around the screen while talking.)

2 R: (0:39) "You can try and use the arrow keys if you want and space to dig."

3 [act]: (0:45) [Luke presses spacebar on the 0 mark, an animation plays of the dog digging on the spot, a green line appears at the fraction location 9/16 and the dog gets sad. See Figure 2a.]

From a mathematical problem-solving perspective, each fraction to be estimated, can be classified as a new local problem. This progress of estimated fractions can be seen on "the chart with the flag" (transcript #1) that Luke identifies, on the left brown bar becoming progressively green on the left of the screen in Figure 2a,b,c.

For the next two fractions shown, 3/7 and 6/13, Luke moved the dog to the end of the number line close to 1, pressed the spacebar, and the dog returned sad again both times. His hasty movements with the dog and space bar presses, leads to the assumption that Luke may still be exploring and trying to understand the game world. This continued to the fourth and fifth fraction (1/13 and 11/12), for which he had kept the spacebar pressed, seemingly experimenting or not being aware, so the dog began digging as soon as the new fractions appeared, both times (transcript omitted and #6). However, due to the fraction 1/13 being close to 0, the dog reacted happily after digging at 0, and 100 bones were awarded. This successful instance led Luke to revaluate his interest and take a more structured approach:

4 L: (1:17) "…so I just found my 100 bones…"

5 NT: (1:18) 5^{th} fraction shown: 11/12, 100 total bones.

6 [act]: (1:18) [spacebar pressed, as soon as the fraction loads, the dog digs, before the sentence on 10 is finished. Dog is sad. Input mistakes like these would happen once in a while amongst participants.]

7 NT: (1:25) 6^{th} fraction: 1/9, 100 total bones.

8 [act]: (1:26) [Luke moves the mouse at around 0.75-1.0 on the number line, but moves the dog to the center 0.45-0.50, and presses spacebar. Dog is sad.]

9 R: (1:36) "So what is happening now?"

10 NT: (1:37) 7^{th} fraction is 9/14, 100 total bones.

11 L: (1:39) "Umm well when you don't get the bones, the dog gets sad, and then there is a green line that shows you were the bone is."

12 [act]: (1:45) [while describing, L moves the dog to 0.05-0.1 and then spacebar is pressed, dog is sad.]

Polya's steps

With this excerpt, Luke completes and verbally confirms Polya's (2004) steps: he shows understanding of his aim to find the bones (#4), makes a plan, then carries it out to acquire them (#8), and reflects on his actions by meaningfully identifying the feedback from the game (#11). First, he mentions that the dog is sad because it did not find the bones, inferring the estimation was not good enough, then he points out the green line (seen in Figure 2b,c,d) confident enough on his hypothesis to proclaim its function, and has thus, in essence, shown full understanding of the aim of the game. Instead of general feedback—bones, happy dog; no bones, sad dog—the green line is a game element that provides corrective feedback: marking where the correct fraction magnitude is located on the number line, as well as simultaneously reflecting how far off the player's actual estimate was. From a local perspective, Luke "fails" at his latest trial (#12). However, after all the local problems he faced, he takes steps towards identifying the global problem.

The local-global problem of *Number Trace*

As in every mathematical problem, players of *Number Trace* need to understand what is asked of them, what are the rules and the aim, as well as how they can interact with the game, in this case how to play it (either successfully or not). At first, through exploration and experimentation attempts are made to grasp the game environment and the rules which occur in it. Participants seem to make assumptions initially, which they then test while playing the game, learning from the game's immediate feedback. This is reflected by Luke's performance on the next items, with the 8th fraction being 3/14:

13	[act]:	(1:55) [the dog is moved immediately to 0.15, and spacebar is pressed, 100 bones awarded, dog is happy.]
14	L:	(1:57) "So 100 bones again."
15	R:	(2:00) "Well done."
16	NT:	(2:00) 9th fraction is 5/13, 200 bones awarded in total so far.
17	[act]:	(2:02-2:07) [3 incremental short steps, where the dog moves and stops, then 1-2 presses to the right, spacebar pressed, dog is sad.] (Please notice how the number of steps is equal to the numerator.)
18	NT:	(2:12) 10th fraction is 3/8, 200 total bones.
19	L:	(2:10) "You also see some fractions at the top" (showing with mouse) "which I think are meant to tell you where the bones are so..." (2:20 as soon as he finishes explaining, he begins bobbing his head.)
20	R:	(2:17) "Mhmmm." (confirming L's comment.)
21	[act]:	(2:13-2:33) [head-bobbing continues for 20 seconds, before L moves the dog at around 0.1 and digs. Dog sad.]

The spontaneous comment mentioning the fractions at the top of the screen (#19), completes the picture of the game's global estimation problem. Asynchronously using Luke's words, the statement is completed after ten fraction estimations: "you see some fractions at the top that are meant to tell you where the bones are (#19), if you don't get the bones, the dog gets sad and there is a green line which tells you where the bones are (#11)".

Emergence of segmentation methods and techniques

In the excerpts above, Luke exhibits two evolving methods: *short-steps* (#17) and *head-bobbing* (#19). At the start, Luke, for the first time and after careful planning, takes an identifiable, intentional approach to solve the problem: takes five steps with the dog, the number corresponding to the numerator of the fraction shown. Using the dog's movement mechanic to steadily interact and measure distance (i.e., the short-steps method) was a common behaviour shown by participants. However, since Luke didn't consider the denominator, his estimation was inaccurate.

Afterwards, Luke exhibits head-bobbing (#19) which, from here onwards, was consistently used throughout his playthrough. After being observed by the researcher for 20 seconds, he is asked about this action:

22	L:	(2:38) "I was trying to count in spaces were the *third eight* (mistakenly called instead of "three eights", #18) was-uh-uhm was like uhm it was on the ground."
23	NT:	(2:41) 11th fraction is 10/14, 200 total bones.
24	R:	(2:47) "Nice! How did you try and count that?"
25	L:	(2:52) "– some lines then I just set them up (shows up and down with finger across the line) to make them into eighths and then. And then I just dug." (moves the dog to estimate this next fraction in-between explanation.)
26	[act]:	(3:02) [moves dog to 0.1, leaves it there unmoving, and then begins bobbing his head counting, for 15 more seconds.]

Two minutes later and after successful digging for 500 bones, Luke adds in relation to his head-bobbing:

27	L:	(5:16) "I counted in my head to make the boxes equal and then I dug where I thought the bones were."

From that point on he used the term "boxes" frequently. Towards the later stages of his session, the fraction was 4/16, when he was asked to talk about the boxes more:

28	L:	(23:45) "So the boxes are basically like number lines, you basically just find a size (second-person perspective), then I test them out (switches back to first-person), I get to the uhm to the number, in this case until I get to 16 and then I would do it back again until I get to 4 so…" (trails off in thought while head-bobbing, pressing space and scoring 300 bones.)

Luke thus reveals a very important insight, showing how a method can become a technique towards solving the global problem: "(…) I get to 16 and then I would do it back again until I get to 4 (…)" (#28). If not phrased so clearly it would be difficult to capture otherwise, with just screen-capturing footage. Participants using a method of segmenting the number line (in this case: first to 16 then to 4) must execute it at least twice, for the relation of numerator-denominator to take effect.

A similar repetition was employed by participants using the short-steps method (#17): Malcom (Mm), aged 13, predominantly made use of this method, taking small, careful steps. However, he took it one step further, leading to a technique dubbed *walking-the-dog*, due its unique three-part execution. Malcom also did a good job in unprompted narrating his decisions and ideas as shown:

29	NT:	(5:54) 2nd fraction is 3/12, 500 bones awarded from the first fraction 7/9.
30	[act]:	(5:55) [three small incremental steps, the third one seems too long, he adjusts slightly to the left. Then goes all the way back to zero to start again (6:00). He takes 12 steps but reaches only up to 0.7 of the number line.]
31	Mm:	(6:11) "Hmm, let's see" [while going straight back to zero (6:15), takes twelve steps again, longer and adjusting each one, till reaching 1 (6:30).] (6:31) "here I try to cut in 12" [while using the short-step method back from 1 to 0, with the same longer and adjustable steps (6:46).]
32	Mm:	(6:47) "nice" [while taking three steps from 0] (6:52) "somewhere here I think." [digs, 300 bones] "nice."

On his next fraction 3/14, walking-the-dog again, he also said while halfway on his first (out of three parts) short-step method: (7:06) "I'm now doing the same tactic"—R: "which is?"—"to separate this in 14 parts". It is interesting to notice how the pattern of taking the same number of steps as the numerator (#30) was exhibited by Luke when using the short-step method (#17). However, instead of digging there and then as did Luke, Malcom resets, tries again and ends up developing a technique he would predominantly use from hereon. Overall, Malcom would walk the dog by using the short-step method three times for each fraction: starting from 0 he would reach 1 with the same number of steps as the denominator. Then, he would take the same number of short steps back, from 1 to 0, to validate his first segmentation. The third time he would walk again from 0 to 1, the number of short steps equal to the numerator, where he would start digging and use his signature phrase "hmm somewhere around here, I think". While walking (during any of these three stages), if he felt one of his steps was unequally spaced, he would frequently retrace it to the one before, and resume walking in his original direction.

All participants were identified as using a segmentation method at least once—especially at their early estimations when first starting the game—and some even talked about it very clearly, such as Max (aged 13) and Ben (aged 15). Both were generally very quiet and would not usually speak unless prompted:

33	R:	(12:22) "What's happening now? How are you feeling?"
34	Max:	(12:29) "Now I'm just trying to think of like.. trying to imagine a kind of bar and splitting it into- whatever the denominator is and then just seeing where I think the bones will be, based on of tha'."
35	R:	(6:19) "Could you tell me what are you generally doing here?"

36 Ben: (6:21) “Well, I see the the given fraction, I try in my brain to- to break up the area in as many parts the denominator says, and then I try to find the numerator in parts.”

Comparable to Luke and Malcom respectively, other participants also sometimes bobbed their heads up-and-down, and every participant took time to adjust their estimations while taking steps when moving their dog to the desired position. In rarer instances, fingers or the mouse were also used in a similar fashion by participants. These infrequent approaches, however, were not repeated sustainably enough and thus were not classified as techniques.

Use of benchmarks

In a more holistic approach, participants used easily identifiable locations on the number line as benchmarks (e.g., 0, 1/2, 1). If a benchmark was identified and considered useful or a fraction was seen as equivalent (e.g., 7/14), participants would override their usually preferred methods or techniques, and immediately move the dog towards the benchmark before adjusting their position to estimate the fraction. For instance, Luke and Malcom would not use their preferred segmentation techniques when they could identify a benchmark: after estimating 6/13 and getting 500 bones, Luke nodded approvingly and said: “…and that one I didn’t really have to use the fractions because 6+6=12 so I just really had to dig around in the middle”, showing a perhaps uncommon connection between 6/12 and 6/13. Furthermore, from all the participants, Malcom was an exception as he spoke while using 1/3 as a benchmark successfully twice: “a bit lower than 1/3” (estimating 4/15), and simplifying 4/12, he mentioned “…this one here I think, on the 1/3”.

Finally, a further interesting use of benchmarks can be highlighted through the gameplay of the second youngest participant: Jeffrey (J) (aged 11), while being quiet about his gameplay choices, showed interest in the game’s elements. For example, he would ask if levels became harder, who designed the game and why, if he can play it after the study. Along with Luke, he expressed interest in the scoring system of the game and made remarks about his own performance, especially noting any increase or decrease in stars between levels. However, his early playthroughs were more successful than Luke’s, as he successfully estimated his first four fractions.

37 NT: (5:44) 5^{th} fraction is 3/12, 1600 bones awarded in total.

38 J: (5:45) “Hmm.” [after a small nudge to the right, the dog is moved confidently to the midpoint stopping upon arrival (5:47). Immediately then (5:48) it is moved to 0.25 (5:49), where, after a minor adjustment, (5:50) the spacebar is pressed. 500 bones awarded, dog is happy.]

Here, Jeffrey was seen moving the dog straight to the midpoint between 0 and 1. This was a common initial approach for him which he had used to estimate the 2^{nd} and 3^{rd} fractions before (i.e., 6/13 and 2/12). For the 5^{th} fraction 3/12, he then goes back to the midpoint of the distance covered (#38), essentially finding half of a half (i.e., one quarter or 3/12). He would use this midpoint-benchmark-segmentation method later again, when estimating fractions that perhaps could be estimated using other approaches. For example, for 1/6, he initially stopped at the midpoint (technically at 3/6) and then took two careful steps towards the left (to 2/6 and then 1/6). We could also classify this midpoint-benchmark-segmentation as being a technique, as it uses the benchmark technique and fulfills the established definition of complexity. However, after the initial midpoint-benchmark step, this method/technique? showed many different variations depending on the presented fraction and the participant. Interestingly, when it was used, it was more often the case for fractions between 0 and 0.5, and not their 0.5 to 1 counterpart. In this regard, Malcom mentioned, while estimating 2/10: "now I am going to go to the middle and from here I will count five, and I think it will be easier", getting 300 bones while doing exactly what he described, and walking the dog from 0.5 towards 0.

DISCUSSION

Methods and techniques in *Number Trace*

Through their playthroughs, between the interplay of local-global problem-solving, participants tried different approaches to estimate fractions successfully. During the process of (and after) understanding the problem posed by the game, participants would use combinations of game elements and their bodies (head nodding, finger pointing) as tools to achieve greater accuracy. Highlighted in the data, either through experimentation or just happening upon a method they liked—or believed was more successful—participants might then develop it further to a technique, which they would stick to, adapt, or abandon over the course of their gameplay.

The techniques used most often by participants can be subdivided into two categories: viewing the fraction componentially—thus segmenting the number line by the denominator then counting based on the numerator—or viewing the fraction holistically—as its relative magnitude between 0 and 1. Identifying that some fractions could be simplified was crucial in the approach selected. While there were only few instances where participants verbally declared they were simplifying a fraction, the speed of their reaction to identifying an equivalent fraction (e.g., 4/12 vs. 1/3) or the proximity to one (e.g., 6/13) nevertheless indicated that participants used this strategy, which is associated with the use of benchmarks rather than segmentation of the number line.

Efficacy: through graceful failure and degrees of success

As mentioned in the chapters above, in mathematical terms, the purpose of *Number Trace* can be seen as asking players to solve the global problem: "Show understanding of fraction magnitude, by successfully estimating fractions on the number line". However, each individual problem the player encounters is also a local problem: "Estimate the position of this specific fraction on the number line".

After playing, and solving a few local problems, participants are encouraged to solve, perhaps unknowingly, the efficacy problem. Combining feedback mechanics and various incentives (e.g., happy/sad dog, bones, stars), players are encouraged to strive for greater accuracy (i.e., closer to the correct location of the target fraction). Looking for ways to collect, for instance, more bones or making the dog happier, should make them keener to pay attention and adapt to the game's rules.

Additionally, a common phenomenon was identified from observing participants' playthroughs regarding the game's degrees of success. Generally, participants showed various levels of success but also frustration, when receiving immediate feedback for their actions, which encouraged experimentation through graceful failure (Plass et al., 2015): due to the various levels of feedback—e.g., animated dog feedback and reward of 0, 100, 300, 500 bones; corrective feedback in the form of the green line; narrative and the star screen shown in Figure 1b after each level— regardless of their estimation, their choice had impact in the game. Interestingly, as participants understood the game more, they began showing signs of disappointment, not only when failing, but when achieving 300 (as opposed to 500) bones. While still learning the game, getting 100 bones for the first time might mean for participants that they succeeded. However, in later playthroughs, this would signify how close they were to the precipice of failure. Similarly, getting 300 bones was not the minimum, therefore, there was nothing to fear or worry about. Nonetheless, it was not the maximum either. Through such feedback dynamics which develop with gameplay experience and performance, participants gradually move towards optimization of their problem-solving approach. For players initially learning the game, the available feedback shows the spectrum of outcomes for their actions. Later, it encourages them be more effective in their estimation attempts.

CONCLUSION

The aim of this study was to evaluate the affordances of using a video game as a problem-solving environment, through the qualitative approach of transcription and analysis of participants' gameplay. Participants played the fraction learning game *Number Trace*. Patterns emerged highlighting the importance of intrinsic integration (Habgood & Ainsworth, 2011) of the learning content and choice of game elements which allowed the facilitation of participants' problem-solving approaches (Polya, 2004). In particular, through the perspectives of local and global problem-solving (Downs & Mamona-Downs, 2007), specific methods participants developed were identified and further developed to techniques for solving the task at hand (Mamona-Downs & Downs, 2004). Participants would make use of in-game affordances as well as physically using their bodies, to assist them in segmenting the number line towards achieving greater accuracy in their fraction estimation. However, if a benchmark (e.g., 0, 1/3, 1/2, 1) was identified as being close to the desirable fraction, participants would favor approaching it holistically, instead of using segmentation techniques. Additionally, younger participants (Luke, Jeffrey) were vocal in their interest for the score and star system, thus perhaps giving insight to the engagement shown in previous studies with young students (e.g., Kiili, et al., 2018a).

Future research, with different age groups, in other settings or with additional data collection measures (e.g., eye-tracking equipment), would be desirable to replicate and further evaluate the idea of conceptualizing a (fraction) learning game as a problem-solving environment to further understand, explore and exploit this idea.

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THE ROLE OF PROCEDURAL AND CONCEPTUAL UNDERSTANDING IN PROBLEM SOLVING

Eszter Kónya[1] and Zoltán Kovács[2]
University of Debrecen, MTA-Rényi-ELTE Research Group in Mathematics Education, Hungary[1]
Eszterházy Károly Catholic University in Eger, MTA-Rényi-ELTE Research Group in Mathematics Education, Hungary[2]

Our paper investigates a problem that can be used to map the conceptual and procedural components of a given mathematical knowledge item. We chose a problem based on the divisibility of powers of ten: finding trailing zeros in the factorial of 100. We evaluated the written performance of 102 students with different mathematical backgrounds, identified four main solution methods, and categorized them according to knowledge types and qualities. Our analysis found examples of both conceptual and procedural knowledge and also identified solutions in which a close unity of both was observed. However, the summary of the results shows that the optimal goal, i.e., the unity of the two types of knowledge, was rarely achieved.

INTRODUCTION

What is the priority in mathematics education? Do we develop algorithmic skills or an understanding of mathematical concepts? The obvious answer is that both are needed. The more difficult question is deciding which types of tasks and methods contribute to both objectives. Incorporating problem solving into a mathematics classroom offers several opportunities to develop conceptual understanding and algorithmic skills instead of focusing solely on routine problems. As Dienes (1973) put it in his well-known book, Building up Mathematics:

> "Children quickly learn conventional answers to conventional questions, which can easily give the impression of understanding. But when asked a less conventional question, it becomes clear that the deep thought behind the words is missing." (p. 25)

Problem solving is where learners are confronted with "less conventional questions," so problem solving can give us insights into the type and quality of understanding. The selection of appropriate problems incorporating conceptual and procedural features of particular mathematical knowledge is crucial in this process.

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In this article, we analyze in detail the solutions to a problem that we have set in several groups of students on several occasions. The focus is on the type and quality of understanding of the mathematical content represented by the problem.

Our research aims to contribute to understanding what knowledge types and qualities are reflected in successful problem solving.

THEORETICAL BACKGROUND

Mathematical understanding, in conjunction with the distinction between procedural and conceptual knowledge and their role in mathematics education, was and still is the central topic of several studies. We mention here only a few key studies on the subject.

Skemp (1976) separated instrumental and relational understanding. He defined instrumental understanding as knowing what to do, i.e., "rules without reasons," while relational understanding means knowing why to do so. He discussed in detail the advantages of relational understanding, one of which we highlight here:

> "It is more adaptable to new tasks … by knowing not only what method worked but why would have enabled him/her to relate the method to the problem, and possibly to adapt the method to new problems." (Skemp, 1976, p. 9)

Therefore, relational understanding is strongly linked to problem solving. Problem solving can help to map whether a student understands the concepts involved in the problem "instrumentally" or "relationally."

The term 'procept,' introduced by Gray and Tall (1994), has also become part of the process-concept literature. A procept is the combination of a procedure, the resulting object, and the mathematical symbol used to represent both. It expresses that mathematical objects and their associated processes form an inseparable unity. Gray and Tall argue that the procepts as cognitive constructs "are at the root of the human ability to manipulate mathematical ideas" and "allow the brain to switch effortlessly from doing a process to thinking about a concept in a minimal way" (Tall et al., 2001, p. 5).

It was presumably Hiebert and Lefevre (1986) who first used the terms procedural and conceptual knowledge. Procedural knowledge means the knowledge of symbols and syntactic conventions, rules, and procedures, i.e., knowledge of specific algorithms. These can easily give the appearance of understanding; see Dienes's quote above. Conceptual knowledge refers to a network in which the relationships are as prominent as the separate pieces of information.

Star (2005) pointed out conceptual knowledge is not always deeper than procedural knowledge. He represented the types and qualities of knowledge in a 2x2 matrix (Table 1) and argued for the existence of deep procedural knowledge.

Knowledge type	Knowledge quality	
	Superficial	Deep
Procedural	Common usage of procedural knowledge	?
Conceptual	?	Common usage of conceptual knowledge

Table 1. Types and qualities of procedural and conceptual knowledge (Star, 2005, p. 408).

The question marks in the table indicate the need for clarification of the corresponding categories. Star proposed considering deep procedural knowledge (DP) as the knowledge of procedures associated with comprehension, abstraction, flexibility, and critical judgment. Moreover, he suggested that superficial conceptual knowledge (SC) can be considered as just a set of concepts, poor in relations.

Baroody, Feil, and Johnson (2007) complemented Star's model and emphasized the link between the two types of knowledge. They distinguished two categories. Adaptive expertise when both conceptual and procedural knowledge is deep because this knowledge can be used creatively, flexibly, and appropriately when solving familiar or new tasks. If at least one of the two types of knowledge is superficial, then it is routine expertise, i.e., it can only be applied in familiar situations.

The relationship between conceptual and procedural knowledge in mathematics is still controversial. Although there is a reasonably broad consensus that conceptual knowledge is a prerequisite for developing procedural knowledge, some argue that the existence of procedural knowledge can also contribute to the development of conceptual knowledge (Rittle-Johnson et al., 2015). They cite several studies (e.g., Baroody and Ginsburg 1986; Canobi 2009; Rittle-Johnson and Koedinger 2009) that reveal evidence that this process is often a two-way street: the development of one leads to the development of the other. These studies focus mainly on the arithmetic knowledge of primary school students, while Friedlander and Arcavi (2012) describe an approach to teaching algebra that integrates conceptual understanding with procedural knowledge in lower and upper secondary school students. The paper acknowledges the traditional approach to teaching algebra that focuses on routine practice and applying rules and procedures but suggests that this approach should be linked to a deeper understanding of the meaning and role of algebraic expressions. In their book, Friedlander and Arcavi (2017) offer practical examples of tasks that promote meaningful learning of algebra and higher-order thinking skills while emphasizing the unity of procedural and conceptual knowledge.

Nowadays, the focus is on developing these two types of knowledge and the relationship between them or other elements of the desired mathematical knowledge (Hurrell, 2021). Figure 1 briefly summarizes the evolution of these concepts.

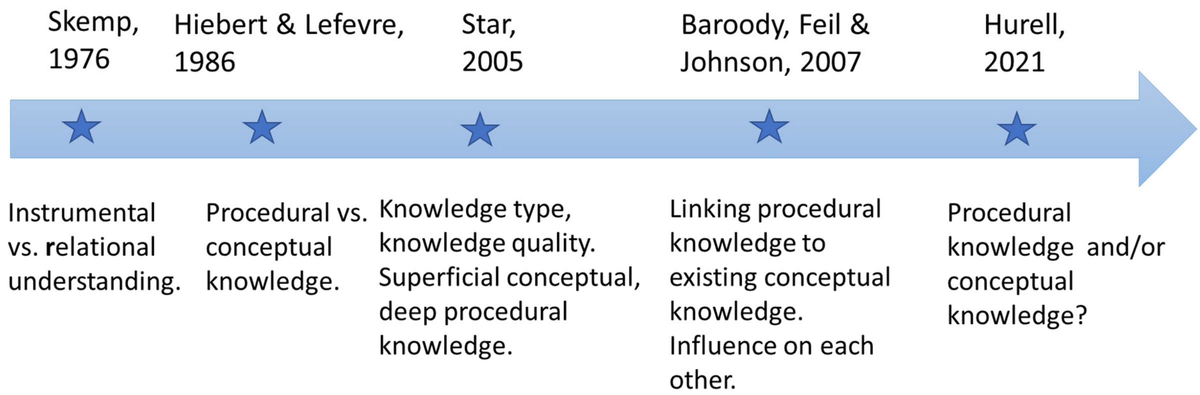

Figure 1. The evolution of the idea of knowledge types.

THE PROBLEM

A few years ago, we found a problem that is a good representation of the conceptual and procedural features of specific mathematical knowledge: divisibility by powers of 10 and the decomposition of natural numbers into prime factors (prime factorization).

Elements of required knowledge and their relationship

Find trailing zeros in 100! (How many zeros does the factorial of 100 end in?)

Understanding the problem means that the solver knows the concept of factorial, namely the factorial of 100, and can link trailing zeros to divisibility by powers of 10. The next step towards the solution is to relate divisibility by 10 to 2 and 5, because a trailing zero appears when a number divisible by 5 is multiplied by a number divisible by 2. If a number has n trailing zeros, it is divisible by 10^n, which means it is also divisible by 2^n and 5^n. Then one should find these factors in the prime factorization of the given number.

To use prime factorization, we need to know the concept and the process that leads to the decomposition. However, it is not enough to have a superficial knowledge of the procedure (e.g., how to decompose a number); we also must know that only the exponents of the prime factors 2 and 5 count, and more precisely, only the smaller of the two exponents. This is true even if the powers of 2 and 5 do not appear explicitly in the solution, but their exponents are determined by counting the number of the relevant prime factors in the product $1\times2\times3\times\ldots\times100$.

The theoretical background of the problem is given by the fundamental theorem of arithmetic, which states that every positive integer greater than 1 can be represented in exactly one way apart from rearrangement as a product of one or more primes. The prime factorization of a given number refers both to the process of decomposition and the product itself, and it is a procept in the sense of Gray and Tall (1994). To obtain the appropriate rearranged form of prime factorization of 100!, it is necessary to add the commutative law of multiplication and the exponential identities to the concept map (Figure 2).

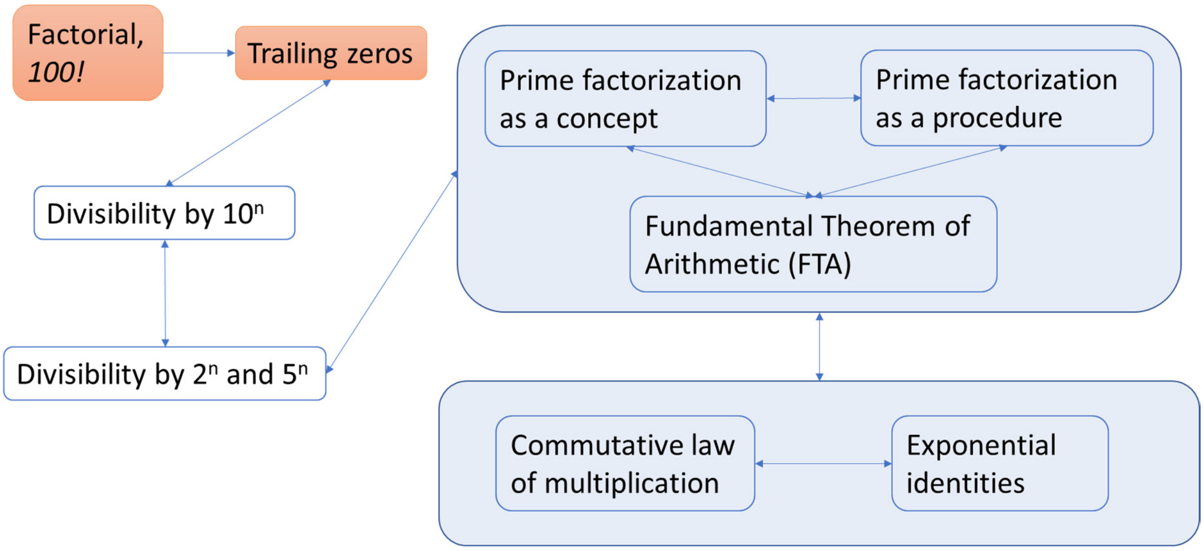

Figure 2. The concept map of the 100!-problem.

The existence of this network obviously requires deep conceptual knowledge. Deep procedural knowledge means that the solver (1) recognizes that the problem is related to the prime factorization; (2) performs only the relevant part of the prime factorization procedure; and (3) provides a way to generalize the problem. Working only with powers of 2 and 5, furthermore choosing the smaller exponent (the exponent of 5) refers to deep procedural knowledge, namely the flexible use of the two appropriate exponential identities: $100!=2^k\times5^n\times\ldots=2^{k-n}\times2^n\times5^n\times\ldots=\ldots\times10^n\times\ldots$ as well as the commutative law of multiplication, because the exponent of 10 determines the number of ending zeros. Although the generalization of the problem was not part of the assignment, the generalizable solution also indicates the existence of deep procedural knowledge. The generalizable solution not only provides an answer to the given 100! problem, but is directly suitable for solving further extended problems, e.g., 1000!, 10^n!

Table 2 gives an overview of the knowledge type and quality concerning the prime factorization:

Knowledge type	Knowledge quality	
	Superficial	Deep
Procedural	Knowing how to decompose an integer and use a given exponential identity.	(1) Connecting the problem to the prime factorization (FTA). (2) Applying the factorization procedure flexibly, elaborating only on the relevant part of it. (3) Providing a generalizable answer.
Conceptual	Knowing separate concepts: prime, product, divisor, …	Finding links between concepts and procedures.

Table 2. Knowledge types and qualities related to the 100!-problem.

METHODOLOGY

Since we both lead problem-solving seminars for student teachers and are involved in designing mathematics problems for high school competitions, we started to investigate how students solve this problem.

The sample was divided into two categories: experienced and less experienced problem solvers. Experienced problem solvers (experts) are 11th-grade high school students who have regularly participated in mathematics competitions for several years. They have considerable experience solving competition problems and are learning mathematics in outstanding secondary schools. Less experienced problem solvers (novices) are teacher trainees who have typically not competed in high school and, as university students, are concerned with higher mathematics. They are familiar with the usual school curriculum, have come from average secondary schools a few years ago, and have only a few courses on problem solving to complete at the university. The number of students participating in the survey is shown in Table 3.

	Groups of students	Number of students
Experts	Math competitors	31
Novices	1st-year teacher students	29
	Upper-year teacher students	42
Total		102

Table 3. The sample.

The students worked individually, where the problem under study was part of a test that consisted of 5 problems for 90 minutes. The survey was organized so that the time factor did not affect the solution's success. Our observations were based on the written products collected, which were analyzed jointly by two expert teachers (the authors of this article).

RESEARCH QUESTION

Based on our general research aim, to contribute to understanding what knowledge types and qualities are reflected in successful problem solving, we have formulated the following research question: What procedural and conceptual knowledge elements of divisibility by powers of ten are present in the answers to the mathematical problem under study? Specifically, are deep procedural and superficial conceptual knowledge (see the question marks in Table 1) present in the students' solutions?

FINDINGS

After carefully analyzing students' written answers to the problem, we identified four main solution methods.

Method 1 (M1). Counting o*nly the tens*

Students sort through the multiplication factors from 1 to 100 and only consider those in which the 10 (or, more precisely, the 0) appears explicitly (Figure 3).

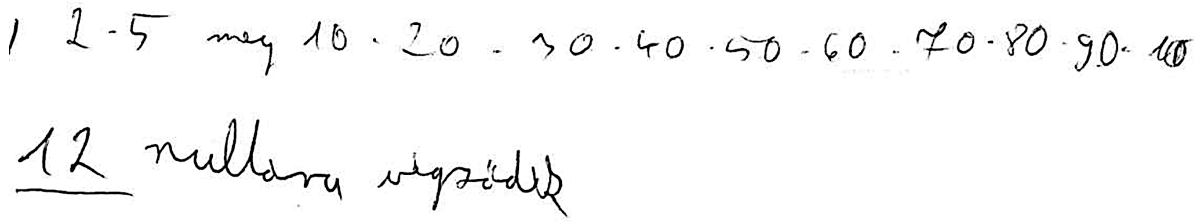

Figure 3. Only the tens count (S3). (Translation: It ends in 12 zeros.)

The possible answers, in this case, are 10, 11, or 12, depending on whether 2×5 and the second zero of 100 are counted. There is no link to prime factorization.

Method 2 (M2). *Finding the pairs whose product ends in zero*

Students use the prime factors of 10, i.e., they investigate divisibility by 2 and 5. They associate an even number with numbers divisible by 5 but ignore the powers of 5 (in this case, numbers divisible by 25). However, these answers suggest incomplete conceptual knowledge because this phenomenon can be explained by students' false assumption that numbers divisible by 2 and 5 can end in one zero only. It also follows that the prime factorization of 100! does not appear in their minds. The possible answers are 20 or 21, like in the solution of S75 that follows:

> Which [adds] 0 to the product: 10, 20; 30; 40; 50; 60; 70; 80; 90; 100. Also, numbers ending in 5 are multiplied by an even factor. Since we have enough even numbers, we only write the numbers ending in 5: 5; 15; 25; 35; 45; 55; 65; 75; 85; 95. That's

20 numbers in total, but 100 = 10×10, so there will be 21 zeros at the end of the number. (S75)

Method 3 (M3). ***Finding pairs of 2 and 5 and remembering that every 25th number has an additional factor of 5.***

Figure 4 presents a typical solution in this category. The student does not refer to the prime factorization but recognizes the case of 25, 50, and 75. The answer is correct, 24 zeros (12 numbers ending in 5 and 12 ending in 0 together make 24 zeros).

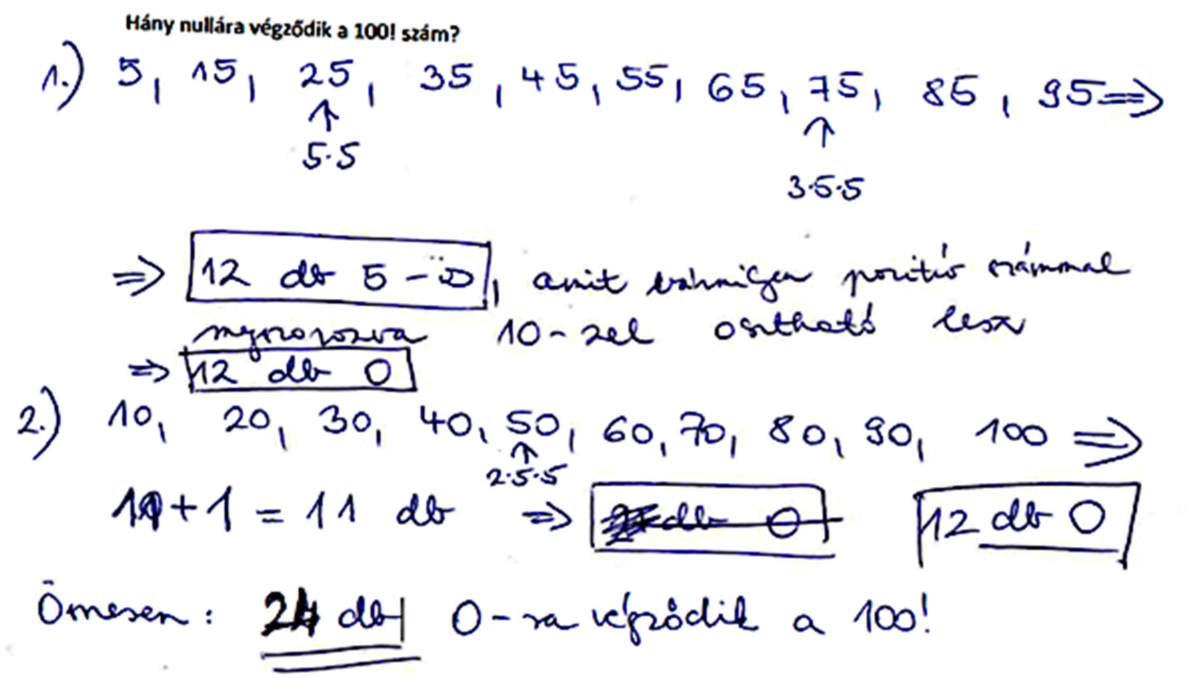

Figure 4. (S51) 12 numbers of 5 multiplied by any positive [even] number will be divisible by 10.

Firstly, S51 looks at numbers ending in 5 that are less than 100 and denotes the prime factorization of 25 and 75. They write: "There are 12 numbers of 5 which, when multiplied by any positive [sic] number, are divisible by 10." Presumably, the student was thinking of even numbers, not positive numbers. From this statement, the student concludes that they have counted 12 zeros. Secondly, S51 marks numbers ending in 0 and indicates the prime factorization of 50. The conclusion is 12 zeros in this section. Finally (in the last line), the student answers by adding the results of the two parts together.

In this correct answer, the idea of prime factorization is already present for some relevant numbers (25, 50, 75) but is not fully implemented for the multiplication 100!.

Method 4 (M4). ***Decomposing 100! into primes and dealing only with the relevant factors.***

This category includes correct answers only, i.e., solutions that result in 24. The solutions differ in determining

M41 just the exponent of 5 without reference to the power of 2 or just a general reference to the fact that 2 has a higher exponent than 5 in the prime factorization,

M42 only the exponent of 5 and indicating that the exponent of 2 is at least 50,

M43 the precise exponents of 5 and 2.

The transcript below illustrates an M42-type answer. It is relatively short but economical because it only determines the exponent of 5 while justifying that 2 has 50 exponents.

> How many 0's are at the end = how many times the 2x5 appears in the prime factorization.
>
> 5: once in every 5th number: 20. Once again in every 25th number: 4
>
> 2: once in every 2nd number: 50.
>
> ⇒ It ends in 24 zeros. (S82)

We present two M43-type solutions; the first determines almost all the prime factors (*Figure 5*), while the second shows an algorithm for determining the number of n!'s trailing zeros (Figure 6).

Figure 5 clearly shows the intention to factorize 100!. After a few trials, the solver realizes that instead of all the numbers, it is sufficient to consider only the relevant ones divisible by 2 or 5. The exponents of these prime numbers are calculated precisely. The method requires some deep procedural knowledge of prime factorization but is not economical because it unnecessarily calculates primes other than 2 and 5. Another shortcoming is that it focuses only on the given problem and cannot offer the opportunity for generalization.

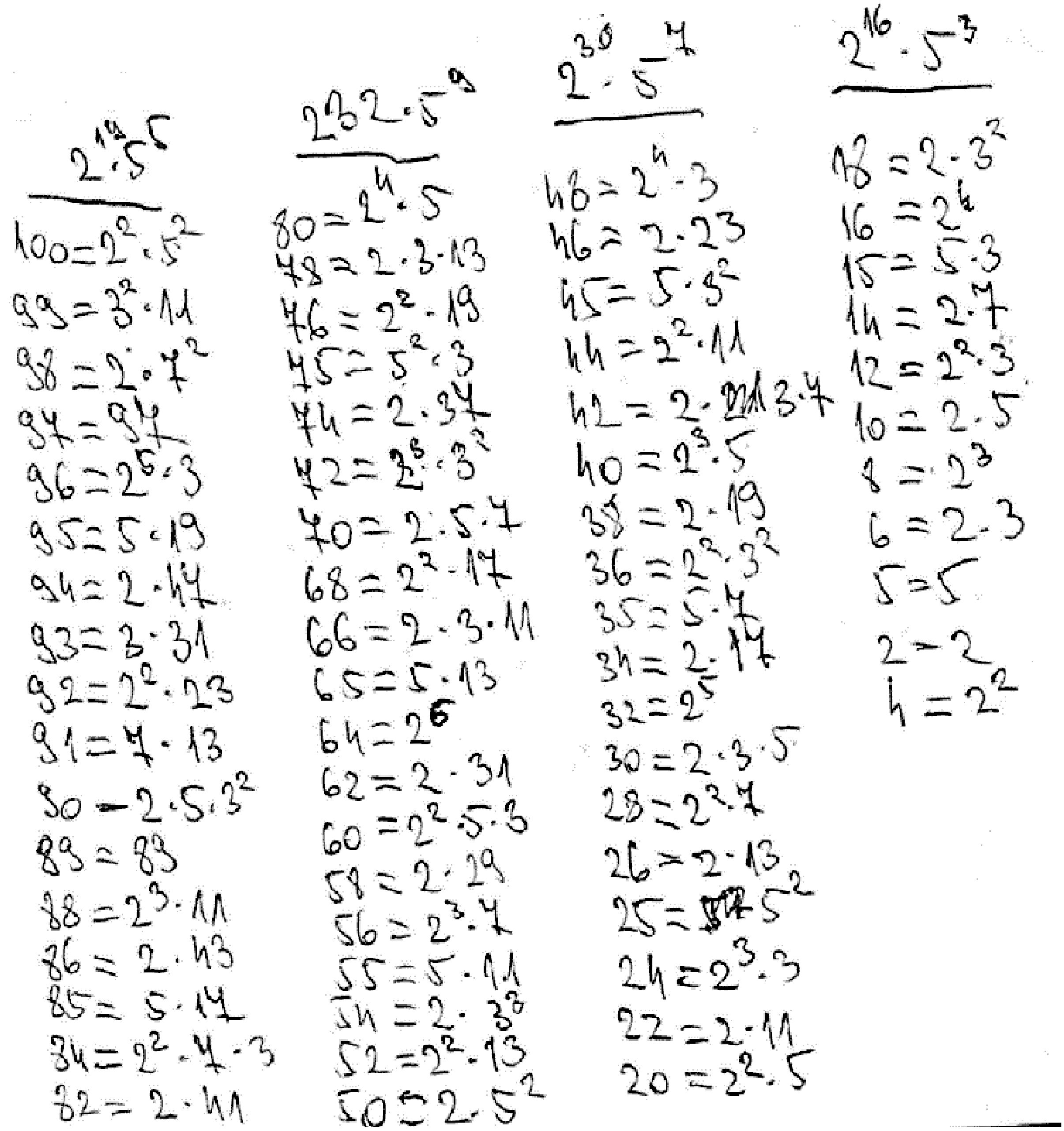

Figure 5. S96 determines almost all the prime factors.

The main idea of the solution presented in Figure 6 is to calculate sums consisting of integer parts of $100/5^n$ and $100/2^n$. The summation goes until the fraction is less than 1. This way, S90 determined first the number of 2's as 97, second, the number of 5's as 24. This solution is algorithmic: all terms in the sum have the same pattern, and the number of terms is controlled. Moreover, the procedure is generalizable; the number of zero endings of k! for arbitrary k can be determined by the summation of integer parts of $k/5^n$, where only finite terms in the summation differ from 0.

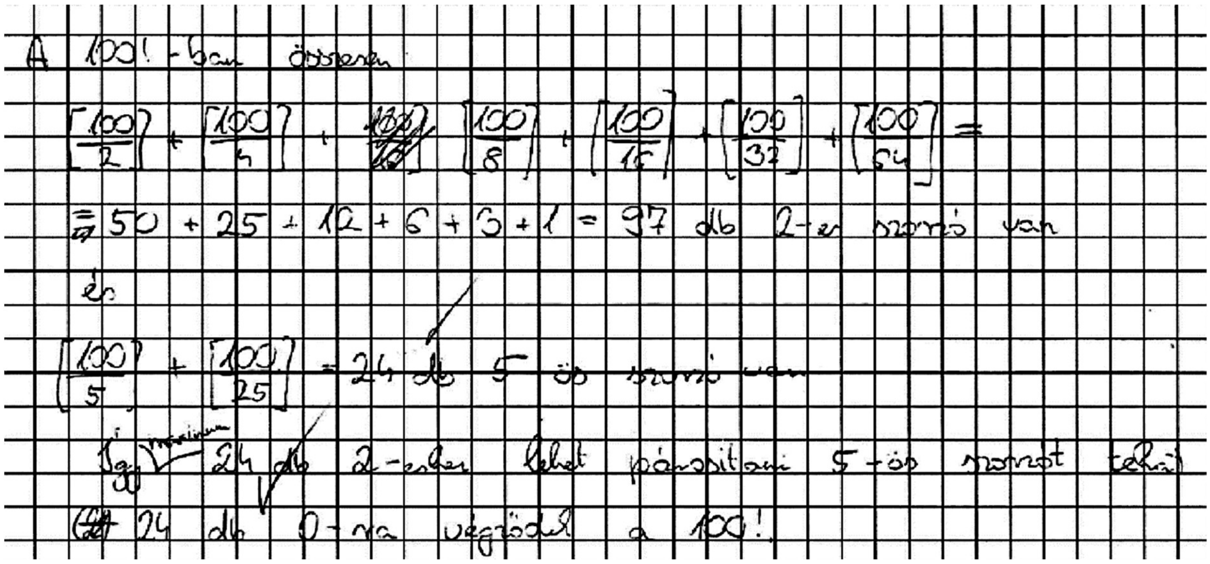

Figure 6. S90 creates a generalizable algorithm. (Translation: Thus, a minimum of 24 2's can be paired with a factor of 5, so 100! ends in 24 zeros.

The solver has created their algorithm based on the prime factorization but uses only those elements of the algorithm that are strictly necessary to get the answer. It has the advantage of being creative, flexible, and easily generalizable. In other words, this method reveals a deep conceptual and procedural knowledge of the mathematical content of the problem.

Table 4 presents the distribution of solution methods by student categories, where M0 means no answer. The table shows that most of the correct answers (M3 and M4) came from the competing high school students, i.e., from the experts. It also indicates that 1st-year teacher students have the weakest results, while upper-year students show a slight improvement.

	M0	M1	M2	M3	M4	Total
Math competitors	4	0	14	1	12	31
1st-year students	15	5	7	0	2	29
Upper-year students	11	9	15	3	4	42
Total	30	14	36	4	18	102

Table 4. Distribution of solution methods by student categories.

DISCUSSION

The conceptual element of the problem is divisibility by powers of ten. This knowledge item is superficial if it only applies to the round tens; perhaps the solution shows the factorization of 10 (2×5) but no longer offers the prime factorization of the powers of 10. On the other hand, we speak of deep conceptual knowledge if the knowledge item correctly includes the rule of divisibility of composite numbers by powers, i.e., the student derives the divisibility by powers

of ten back to appropriate powers of at least five, explicitly mentioning or tacitly acknowledging the role of powers of two.
The procedural part of the problem is to carry out a divisibility analysis corresponding to conceptual knowledge. Here, we can hardly talk about implementing a procedure for a solution based on round tens because the concept itself directly contains the answer (round tens are obviously divisible by ten). If the student argues the divisibility by powers of 5 by direct counting, we are talking about superficial procedural knowledge. Procedural knowledge is considered deep if the student performs the prime factorization in a flexible and generalizable manner.
Table 5 shows the identified solution methods placed in Star's table (Table 1).

Knowledge type	Knowledge quality	
	Superficial	Deep
Procedural	M3	M4
Conceptual	M1, M2	M3, M4

Table 5. The identified methods are placed in Star's table.

As mentioned above, the procedural knowledge under consideration is absent in the M1 and M2 methods. This is even though superficial procedural knowledge is assumed to be present in all students since the prime factorization algorithm is well-known and often used in mathematics classes. Likewise, in M1 and M2, we found examples for the question mark indicating in Table 1 a superficial conceptual knowledge.
M3 is considered an indication of deep conceptual and superficial procedural knowledge. The concept of division by powers of ten is well understood and used by these students. Although the idea of prime factorization for 25, 50, and 75 appears, it is not fully developed concerning the complete solution.
We further investigate the fine structure of M4 solutions to reveal deep procedural knowledge about prime factorization. In Table 6, students are divided into two categories (expert and novice), while their answers are into three.

Students employing M4 know that the solution depends on the product of 5 and 2 prime factors. However, one difference is whether and how they justify that there are more 2's than 5's: (1) dealing with the exponents of 5 only; (2) calculating the exponents of 5 with reference to exponents of 2; (3) exact calculation of the exponents of 5 and 2. Therefore, (2) is considered more economical than (3). At the same time, the precise determination of the exponents of the two relevant primes, if they are not based on a detailed calculation but on the construction of an algorithm (Figure 6), offers the possibility of generalizing the problem.

	(1) Exp. of 5 only	(2) Exp. of 5, reference to exp. of 2	(3) Exp. of 5 and 2	Total
Experts	3	3	6	12
Novices	3	3	0	6
Total	6	6	6	18

Table 6. Categorization of M4 answers.

It is clear from Table 6 that experts, i.e., competitive high school students have a higher proportion of this deep procedural knowledge than novices, i.e., teacher students. Nearly half of the experts linked the problem to prime factorization, while far fewer of the novices.

Only 6 students from 102 (all experts) have succeeded in creating a generalizable counting algorithm. Although the generalizability of the solution was not expected, it was considered a sign of deep procedural knowledge. Third-year teacher students have not reached the expert level despite participating in problem-solving seminars at the university. However, we should mention that superficial procedural knowledge can also lead to success in this problem, as seen in the correct answers obtained using the M3 method (see Table 4.

SUMMARY

Mapping the concepts' network and focusing on the characteristics of procedural knowledge confirmed previous research findings (Star, 2005; Hurrell, 2021) that the presence of conceptual and procedural knowledge elements, together with the relationships between them, leads to successful problem solving. Since the 100! problem does not refer explicitly to the principle of prime factorization, proved to be suitable to answer our main research question (What procedural and conceptual knowledge elements of divisibility by powers of ten are present in the answers to the mathematical problem under study?). All four categories (SP, SC, DP, DC) of knowledge types and qualities appeared in students' answers. The result allowed us to fill out the "question mark cells" (DP and SC) in Star's table (Table 1) with examples.

Finding examples of deep procedural knowledge is always more challenging than finding examples of superficial conceptual knowledge. Therefore, we summarize only our results on deep procedural knowledge related to prime factorization (Table 4).

Only 18 students of 102 realized that they needed the prime factorization to solve the problem, 40 tried to match numbers divisible by 2 and 5, and 14 only looked for the tens. The remaining 30 students did not understand the problem at all.

The 18 students who used prime factorization (M4) knew the factorial concept and the multiplicative nature of divisibility. Thus, they performed the decomposition on the factors of 100! and not on its calculated value.

Four students answered correctly despite not using the prime factorization method (M3). Their procedural knowledge of prime factorization is considered not deep, but their conceptual knowledge is.

Overall, we can conclude that at least the conceptual knowledge must be deep to solve the 100! problem correctly. However, deep conceptual (rich in relationships) and deep procedural knowledge characterized the most successful solutions. We have found that indicators of deep procedural knowledge can be generalizability, flexibility, and economy.

The fact that teacher students were less successful than the competing high school students may suggest that deep procedural knowledge, as well as deep conceptual knowledge, is not the focus of contemporary mathematics education in Hungary nowadays. Teacher training may be helpful, but it is still insufficient to develop deep conceptual and procedural knowledge to the required level.

Acknowledgment

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HOW STUDENTS CONTROL THEIR WORK IN MATHEMATICAL PROBLEM-SOLVING PROCESS

Márton Kiss[1] and András Ambrus[2]

University of Debrecen, Hungary[1] and Eötvös Loránd University, Hungary [2]

Abstract: In the mathematical problem-solving process critical issues are control, evaluation and reflection, that are metacognitive activities. In our paper we analyse a case study Hungarian secondary school students work, we describe the experiences investigating these activities' control, evaluation and reflection. In one mathematics lesson of the class of 31 students, an independent problem-solving exercise was carried out. After analysing the students' individual work, we selected 4 students to be interviewed about the problem we had set. Nearly a third of the class tried to reflect overall solving process. The students are not used to critically evaluating their own thoughts and actions. Experience has shown that successful problem solvers being better at metacognitive activity than unsuccessful solvers.

INTRODUCTION

The evaluation and the control our actions and decisions are important human characteristics and play an important role in everyday life, too. In the school, we try to educate our students about self-evaluation, self-control. Mathematics as subject is suitable for educating disciplined, critical thinking. This is a nice idea, but what is the reality in the mathematics classrooms? Based on our several decades of mathematics teaching experiences in Hungary, we may state, most of the students do not like evaluating critically their work at solving mathematical problems. The Looking back phase by Pólya (1977) is important, but it is an internationally neglected part of mathematics problem-solving teaching (Ohlendorf, 2017; Mason, 2022). Our aim is to examine how Hungarian secondary school students evaluate and control their problem-solving activities. We were curious to see what the situation was in terms of our research question in an incoming ninth grade class that would be studying mathematics in an advanced class. What characterises the problem-solving control, evaluation and reflection activities of the students in our case study, who were enrolled in a mathematics extra-curricular class? In one mathematics lesson of the class of 31 students, an independent problem-solving exercise was carried out. After analysing the students' individual work, we selected 4 students to be interviewed about the problem we had set.

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THEORETICAL BASE

Students' mathematical problem solving is a very complex phenomenon. It can be considered as a nexus of cognitive, metacognitive, affective, instructional, environmental and cultural attributes, at the least (Barkatsas & Hunting, 1996). In this article we focus on the metacognitive factor.

What is metacognition?

Flavell's (1976) definition seems to be generally accepted, having incorporated two important aspects of metacognition, monitoring and regulation of one's own cognitive processes:

> "Metacognition refers to one's knowledge concerning one's own cognitive processes and products, or anything related to them, e.g., the learning-relevant properties of information or data. Metacognition refers, among other things, to the active monitoring and consequent regulation and orchestration of these processes in relation to the cognitive objects on which they bear, usually in the service of some concrete goal or objective." (Flavell, 1976, p. 232)

Metacognition is a key predictor of performance in learning and problem solving. Previous studies have shown that participation in metacognitive activities was positively associated with the use of all relevant strategies. Problem-solving strategies were found to be a fully mediating variable between metacognition and performance (Sperling et al., 2004; Zhao et al., 2019).

Individual differences in metacognitive skills seem to be even more important for secondary school students. In a recent study of older schoolchildren, Veenman (2006) investigated the contribution of metacognitive skills and general intelligence to the development of mathematical learning achievement. Overall, the results showed that both intelligence and metacognitive skills influenced mathematics achievement. Interestingly, metacognition outperformed intelligence as a predictor of mathematical learning performance. Although it was related to metacognition, intelligence did not play a significant role in the mathematics learning performance of secondary school students.

Verschaffel (1999) also pointed out that metacognition is particularly important in the process of mathematical problem solving. He emphasizes the importance of metacognition in the sense of evaluation in the final stage of mathematical problem solving when computational results need to be verified.

Some difficulties in mathematical problem solving are mostly associated with poor metacognitive activities. These include the lack of control processes considered essential for successful problem solving (Lester, 1985; Silver, 1985 Schoenfeld, 1985, 1987).

According to Schoenfeld (1987), control processes during problem solving involve: monitoring the work being done and using the input from these observations to guide problem solving operations.

Mason et al. (2010) highlights that mathematical thinking can be improved through practice and reflection, and the time spent thinking about problems and trying multiple approaches is worth it.

Depaepe et al. (2010) have shown that teachers rarely or not at all pay attention to the "how" and "why" of using a metacognitive skill. Dignath and Büttner (2018) confirmed that teachers teach mainly cognitive and very few metacognitive strategies. The teaching and application of Pólya's (1977) problem-solving model could provide a complex opportunity to incorporate metacognitive strategies.

The model of Pólya

Pólya (1977) did not mention the word metacognition, but in his problem-solving phases we can find most of the essential metacognitive activities. No wonder that in Hungary Pólya has influenced the mathematical problem-solving teaching in a great manner. We mention only some questions and advice from Pólya for example.

(1) Understand the problem: Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
(2) Make a plan: This phase is decisive at finding the solution idea. Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. Do you know a related problem? Do you know a theorem that could be useful? Have you considered all the essential notions involved in the problem?
(3) Carrying out the plan: Carrying out your plan of the solution, check each step. Can you see clearly that the step is correct? Can you prove it is correct?
(4) Looking back: Examine the solution got. Can you check the result? Can you check the argument? Can you derive the result differently? Can you use the result, or the method, for some other problem?
In our case study, we focused our analysis primarily on the verification of the result and the argument from the Pólya's phase of retrospection.

Schoenfeld's categories of knowledge and behaviours during problem solving

Schoenfeld described five categories of mathematical knowledge and behaviour:

- Resources (the knowledge base);
- Heuristic (problem-solving) strategies;
- Control (Monitoring and self-regulation, aspects of metacognition);

- Beliefs;
- Practices (the consistent activity patterns of a particular intellectual or other community) (Schoenfeld, 1992)

We describe only the control and metacognition category in detail, because in our research we looked at these.

Control and metacognition, including planning, monitoring and assessment, decision-making and conscious metacognitive acts. Decision making, self-regulation, self-evaluation, self-control belong to metacognition. Cognition is involved in doing, whereas metacognition is involved in choosing and planning what to do and monitoring what is being done. Research on metacognition shows that successful problem solvers can reflect on their problem-solving activities, have available powerful strategies for dealing with complex and unknown problems, and regulate (even subconsciously) powerful strategies efficiently. Novices, in contrast, have gained fewer problem-solving strategies, are less aware of the utility of them and do not use them effectively in the acquisition of new learning (Schoenfeld, 1992, 2015).

A Hungarian mathematics textbook Grade 6

In this textbook (Csordás et al., 2007), there is a chapter "How shall we solve problems?" in the spirit of Pólya (1977), with practical suggestions for teachers and students, aiming to help the students be conscious at solving problems. Unfortunately, a lot of Hungarian practicing teachers do not realise this chapter in their teaching and use the allowed time for consolidation of concepts, skills, procedures.

RESEARCH QUESTION

What characterises the problem-solving control, evaluation and reflection activities of the students in our case study, who were enrolled in a mathematics extra-curricular class?

Few studies have investigated the metacognitive activities of Hungarian secondary school students in mathematical problem solving. We wanted to contribute to the research in this area with our case study.

METHODOLOGY

Our case study involved students from a small Hungarian town secondary school. In our research, we analysed the problem solving of a group (31 students) of ninth grade students. Within Phase 4 (Loocking back) of the Pólya's model, the focus of our research were the control, evaluation and reflection of ideas, results and arguments.

Our sample was an incoming class of ninth-grade high school students (31 persons) enrolled in an advanced mathematics class. The students came from different schools, with different teachers, backgrounds, and habits. We wanted to know what the situation is with our research question in an incoming ninth-grade class that will be studying mathematics in an extended timetable. We brought them a worksheet in the 2nd week of the school year, so the increased timetable was not yet in force at the high school. The task sheet included a combinatorial task with specific assignments adapted to the questions of the problem-solving model of Pólya (1977). For example, when deriving the answer, we asked for a justification of the steps, followed by a reflection on the answer ("Examine your answer. Check. Are you sure you have the right answer? Why?"). In this way, we intended to bring Pólya’s stage 4 to life, which is an often-neglected phase during the problem-solving process (Ohlendorf, 2017; Mason, 2022,). Students have not received specific training on the instructions we have planned. We had no contact with the class before and after the research.

The problem: 2006 is a number where the digit in the lower place-value position is three times larger than the digit in the higher place-value position. How many of these positive integers are less than 2000?

The solution: Considering that the higher and lower place-value are pairs of (1; 3), (2; 6), (3; 9), and that we are looking for numbers less than 2000: 3 two-digit numbers, 30 three-digit numbers, and 100 four-digit numbers, there are 133 numbers that satisfy the criteria.

We wanted to give the students a task that could be solved with common sense and without any special knowledge. On the other hand, we have also tried to ensure that the verification of the solution does not require a rote procedure. A combinatorial task is suitable for this purpose, which has the difficulty of lacking well-established and reliable verification strategies (Mashiach Eizenberg & Zaslavsky, 2003). We therefore expected that the students' work would reflect their individual attitudes and knowledge of control, evaluation and reflection. When analysing students' works, their answers were grouped into three categories.

- Correct answer: the solution was complete, and the result was correct
- Incomplete answer: the total was mathematically correct, but either did not consider all cases (e.g., only four-digit numbers were considered) or did not consider that numbers greater than 2000 were not part of the answer
- Incorrect answer: the interpretation of the problem, the idea or the calculation was incorrect

After solving the problem, we tried to elicit students' reflections with instructions. We expected them to check whether they had solved the problem exactly under the condition that was given, to check the steps of the solution, to reflect on their idea, and possibly on the reality of the answer.

We developed inductively the categories of reflections (in short we refer to controls, evaluation and reflection). We identified the following categories of student reflections:

- Complete reflection: written reflection on the whole solving process (e.g., "in four-digit cases, you can write the numbers between the two digits [first and last] from 00 to 99 (that is 100 digits); in three-digit cases, you can write any digit from 0 to 9 for each pair of digits according to the previous principle, here the condition below 2000 already allows three pairs of digits; in two-digit cases, there is no intermediate part, so here the three pairs of digits give three numbers.")
- Incomplete reflection: written reflection on some steps of the solving process (e.g., "because three times a digit greater than 3 is already a two-digit number" — suggesting why at most 3 can be a three-digit number and two digits in the largest local number case)
- Sloppy reflection: this includes those who either listed only a few examples of the numbers they were looking for; redid a calculation they had done before; or just wrote "Correct because I tried all the possibilities."
- No reflection: no

Note that students' reflections were examined only based on their written work. We have no information about students' reflections that may not have been written down.

After analysing the students' individual work, we selected 4 students to interview about the problem we had set. We selected students who could not solve the problem and made typical mistakes. We wanted to use the interviews to observe how the incorrect or incomplete answers were resolved. We had the worksheet in front of the pupils and asked them orally about their answer. All our interviewees were asked to read the text of the exercise again and to explain their answer verbally. If they did not realise the mistake on their own, we asked them questions to help them to find the right answer.

RESULTS, EXPERIENCES

One of the 9th grade students answer shown (see Figure 3) is an almost perfect picture of the level of detail we would expect from students. Figure shows the student has attempted to justify and detail the steps. A few other students gave such detailed answers, but the most of them did not.

The pupil (see Figure 3) looked separately to see how many two-digit, three-digit and four-digit cases there were in the given conditions. He was careful to check positive integers less than 2000 and to check that three times the digit in the higher place value is in the lower place value. And for intermediate local values, all digits from 0 to 9 were considered.

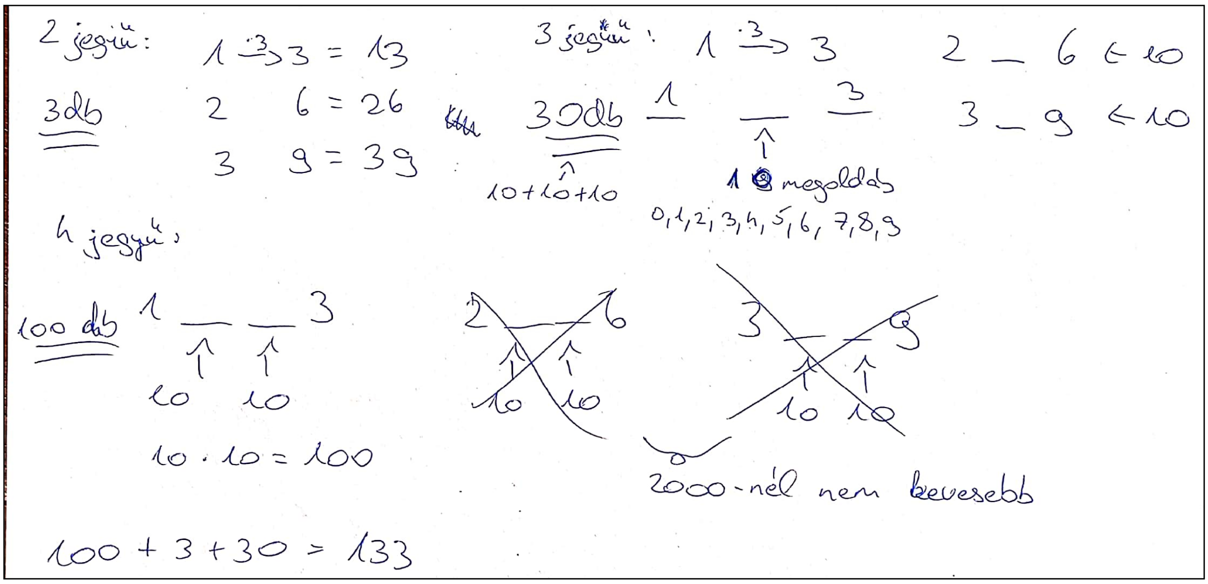

Figure 3. A correct answer for the combinatorics problem.

The table summarises the relationship between the correctness of students' answers and the quality of their reflections (see Table 1).

Reflection	Correct answer	Incomplete answer	Incorrect answer	Total
Complete reflection	4	5	2	11
Incomplete reflection	0	3	3	6
Sloppy reflection	3	4	3	10
No reflection	2	0	2	4
Total	9	12	10	31

Table 1. Grouping of students according to the quality of answer and reflection.

The analysis of the written work shows that all but 4 people have done a reflection. Almost one-third of the 31 people tried to reflect on the overall solving process. Of those who solved the problem incorrectly (10), only 2 students tried to reflect on the overall solving process, while the proportion was higher for those who solved it incompletely and correctly. (Almost half of the latter two categories tried to summarise and reflect on the whole solving process in writing.) However, even the most correct solvers did not comprehensively reflection their answer. (So that all conditions and the whole process are reflected.) None of those who solved the problem incompletely or incorrectly reflected overall solving process but simply re-explained it and did not detect the error (e.g., omitted cases or errors of principle).

That is, students who solved the problem more successfully than all their peers were observed to be able to reflect more correctly and more competently than their peers. An explanation for this experience may be what Schoenfeld (2015) mentions. Research on metacognition has shown that successful problem solvers are able to reflect on their problem-solving activities and effectively regulate (even unconsciously) the strategies available to them.

In the written work, there was not a single student who commented that they were unsure about their answer, questioned it, or even admitted that they had to change their answer afterward. This may be because a combinatorial task usually does not have a well-established and reliable control strategy (Mashiach Eizenberg & Zaslavsky, 2003). The lack of a specific checking protocol for this combinatorial problem did not facilitate the reflection. Therefore, the checking required a more intuitive approach from the students.

Reasons for selection of the students (S1, S2, S3, S4) and highlighting the point of the interviews

S1: Did not consider condition less than 2000; 0 was also included in the largest local value; did not consider two–digit case. As a check, he gave specific examples which included numbers greater than 2000. During the interview, the student noticed his mistakes immediately when we asked him to reread the text of the exercise.

S2: Only 4-digit cases were considered; for intermediate local values, only 9 possible digits were considered. His check consisted of a single division. Difficulty in getting the interview to lead to a solution. S2 said in the interview that he had little time for the check and the other aspects, because he first deduced the answer to the problem and only then he went on to the other parts.

Interviewer:	What was the reason for filling in the exercise sheet this way?
S2:	I think my priority was solving it and I would get more points [if the worksheet would be evaluated].

S3: The justification for the answer was lacking. Did not consider the relationship between the first and last digits. On checking, returned only to the condition less than 2000. When giving examples, he correctly substituted only the correct pairs of numbers. At the interview, he finally realised what he had done wrong after several helpful questions.

S4: He was very careful in describing the data and the derivation of the answer for the problem. Nevertheless, he made mistakes in his answer because he included cases that did not correspond to the relationship of the digits. As a check, he gave specific examples which otherwise satisfied the relationship between the digits, but which included numbers greater than 2000. During the interview, he realised his mistake after asking helpful questions.

These four examples also confirmed our experience that, in most cases, the students' reflection was superficial and forced. They are not aware of the importance of this, and it has not become part of their habit to redefine and review the problem and its answer. From the experience of the interviews, it was common that all the students managed to find the solution, which was most helped by re-reading and re-interpreting the text of the problem.

SUMMARY AND CONCLUSIONS

The aim of this case study was to gain an insight into the control, evaluation and reflection activities of 31 Hungarian ninth-grade secondary school students we studied during independent problem solving. This insight was gained through a combinatorial problem solved independently in a classroom. The problem sheet we submitted included instructions for students to reflect on their answers.

Experience shows that the first written thought dominated the choice of idea or strategy to solve the problem. There is no indication from the students' work that anyone replaced or critically considered the idea or strategy that emerged. In retrospect, as a reflection, they almost only re-explained their previous thoughts. This is not to be underestimated, as it does not exclude the possibility of discovering a mistake. The pupils are not used to critically evaluating their own thoughts and actions, probably because this is not how they were socialised in school and mathematics lessons. Nearly a third of the class tried to reflect overall solving process. This result is commendable and can be built on, improving the quality of reflection and critical thinking.

We believe that the experience of successful problem solvers being better at metacognitive activity than unsuccessful solvers in the Hungarian students we studied confirms the relationship between metacognition and performance (Sperling et al., 2004; Veenman, 2006; Zhao et al., 2019).

In the interviews with ninth grade students, there was the experience that the teacher's questions or the re-reading the problem were able to lead the students step by step to the cause of their mistake. The students we interviewed lacked the metacognitive strategies to independently discover when they had made a mistake (Depaepe et al., 2010; Dignath & Büttner, 2018). In addition, the interviews also showed that the selected 4 students are not used to and do not feel the importance of the control, evaluation and reflection activities.

The case of the class in this paper also shows the need for teaching students to look back on their own work. It seems to be a greater challenge than transferring subject knowledge. In our opinion, one way to achieve this goal is for the teacher to emphasise the importance of reflection through long-term and regular example and to require consistently.

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CAN MOVEMENT-ART LESSONS SUPPORT THE LEARNING OF MATHEMATICS IN ELEMENTARY SCHOOL?

Gabriella Tóth-Babcsányi

University of Debrecen, Doctoral School of Mathematical and Computational Sciences, Debrecen, Hungary

Szekszárd Waldorf Elementary School and Kindergarten, Szekszárd, Hungary

In this report, I will present action research involving two teacher-colleagues from a reform pedagogy school in Hungary. The topic was the fractions in mathematics and rhythmical and musical exercises in movement during eurythmy lessons. The question was, could children in movement-art classroom develop arithmetical thinking and problem-solving. I answer this question through a qualitative analysis of the activities under study.

INTRODUCTION

Exploring possibilities that lie in the interdisciplinary form of learning and teaching was the initiator of this pilot research. How can we link movement-art and handcrafts tasks with mathematical concepts and develop students' problem-solving competence? The school, in which the research was carried out, professes the principles and methods of one of the reform pedagogical trends that emphasize the importance of aligning the contents of different subjects and the role of artistic activities in the learning process. Burkholder (2019) refers to the two-way relationship between mathematics and arts, namely that mathematics creates artistic works, just as artistic works help to develop mathematical understandings. By studying movement-art tasks, for which solution students display spatial forms, as well as handcrafts lessons tasks, similarities with the geometric representations of the concept of fractions were obvious, thus in this research the exercises were created with the intention to develop pupils' problem-solving competence in the process of getting acquainted with the concept of fractions.

THEORETICAL BACKGROUND

According to Dienes (1999) abstraction can only be achieved from concrete things, and to be able to abstract well, one must become familiar with a wide variety of concrete things. In his opinion, we can most successfully escort children to abstract level with a very specific starting point by giving them enough distinctive and varied experience. At the beginner's level, in case of young children, this means very genuine sensory-movement experience, manual activity with movement and touch that generate willpower.

In I. Papadopoulos & N. Patsiala (Eds.), *Proceedings of the 22nd Conference on Problem Solving in Mathematics Education - ProMath 2022* (pp. 125–129). Faculty of Education, Aristotle University of Thessaloniki.

Servais & Varga (1971) claims that as only few opportunities are offered to students during their school years to experience and make them feel and enjoy the beauty of mathematics [...]

> “pupils must be given sufficient opportunity for free, playful, creative activity where each can bring out his own measure of wit, taste, fantasy and display thereby his personality. [...] An important type of problem developing (and making use of) the sense of beauty is the search for patterns. (p. 16.)” (Servais & Varga, 1971)

As Watson (2005) states,

> “rather less obviously, teachers can exploit classical rhythms to develop a sense of fractions, as musical notation does in time signatures and note values. The added feature of dance can be used to show students that they know these relationships already through their movements, through their beating out of rhythms, so that fractions express what their bodies can already do (pp.19 - 20)”.

According to Goddard-Blythe (2015) senses play an important role in learning, human body itself is a big receptor, a means of receiving information. Also, she highlights that movements are crucial for learning because if a child cannot develop his balance or the automatic regulation of movements, it could set his learning back.

Waldorf pedagogy, which belongs to reform pedagogy trends to renew pedagogical thinking and educational practice in a child-centred way (Nemeth & Skiera, 2003), was developed by Rudolf Steiner in 1922. Eurythmy is the name of a movement art inspired by Rudolf Steiner and Marie Steiner- von Sievers - in the early 1920s (Steiner, 1984). In the 4th grade, in accordance with children’s age characteristics, eurythmy supports learning the C Major scale, the 1/4, 2/4 and 3/4 times, and musical sounds and melodies through movements. As a connection to mathematical contents, learning about the principles of music, help children to acquire mathematical concepts and problem-solving competence. (Curriculum for Hungarian Waldorf Schools, 2020).

METHODS

In this paper, pilot research is presented that was composed and accomplished by a teacher of mathematics, a teacher of eurythmy, a class teacher and it involved 16 pupils from the 4th grade in Waldorf School in Szekszárd, Hungary in 2021/22 school year. The research question was: could children in movement-art and handcraft lessons develop arithmetical thinking and problem-solving? The topic was the introduction of the notion of fractions. In the initial, planning process three tasks were defined. As a second step of the research, the implementation of these tasks as well as the data collection via notes, interviews and photos were carried out. The third step was the analysis and the evaluation of the collected data, and the fourth step was the further consideration of related issues.

RESULTS

In the first task, during movement art lessons, factions were represented by spatial forms that appeared when students performed the polonaise dance (Polish dance). That was followed by drawing the choreography patterns. Thereby, certain geometric figures appeared (circle, triangle, rectangle etc.), which represented the unit, whereas their parts represented fractions, for which marking symbols were introduced, see Figure 1.

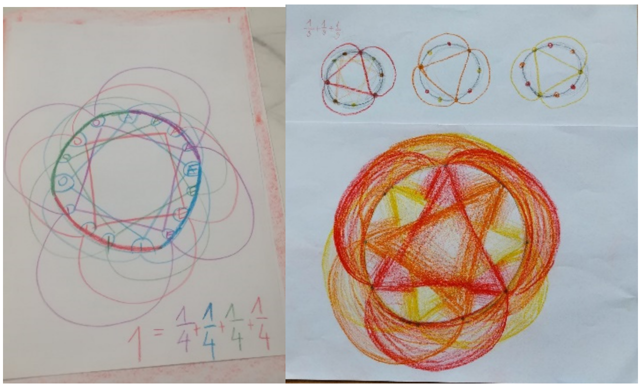

Figure 1. Students' work

The problems of the second task could be solved by performing a dance, where the emphasis was on the area covered by pupils' steps, taking into consideration the musical pieces - used as accompaniment - in which the rhythm was determined as 2/4, 3/4, and 4/4 times. In the exercise book images of the covered area were drawn and colored. The unit was represented as an area of a rectangle, and fractions as part of that area. The third exercise was the combination of spatial movements and clappings, the related drawing task was the graphs that showed the relation and ratio between steps and claps. The unit was represented as a section and fractions as a part of the section.

In handcrafting lessons, the tasks consisted of designing patterns on the fabric surface, by representing certain fractions, elaborating by using the cross-stitch technique and asking questions related to relations, compatibility, and fractions representation, see Figure 2.

Figure 2. Students' handcraft pieces of work

To answer the research question, tasks were created with the aim that students could experience various representations of fractions and develop their problem-solving competence, as well as they practiced how to ask questions about the problem situations, how to plan a multi-step sequence of actions, and solve inference tasks. Summing up the results of the interviews it can be said that the students were interested throughout the lessons. The majority stated they waited with interest for the lessons and were excited and keen on solving tasks, more of them emphasized that the drawing and colouring tasks were very enjoyable, whereas the handcraft tasks were relaxing for them. Some of them claimed that the instrumental music that supported the movement-art lessons, helped them to concentrate. As a result, students successfully solved exercises on tests and homework they were given related to fractions. The teachers' reflections revealed that lesson-planning was more conscious and "qualitative". They were more confident and saw it as an opportunity to "change perspectives", experience the mutual influence and impact of subjects and teaching topics, and take advantages of the potential to work with teacher colleagues. The fourth step of the research was the exploration of the possibility of a continuation. Beside the fact that it may be tested for a larger number of students, teachers saw the possibility to connect the development of other mathematical concepts with movement art and handcraft tasks. For instance, the teaching of angles could be supported by the movement art tasks related to the intervals between musical tones.

CONCLUSION

The research question was: Could children in movement-art and handcraft lessons develop arithmetical thinking and problem-solving? During the research, it was possible to find a connection between the concept of fractions and the movement and handcraft tasks, and the diversity of representations and the related issues that appeared helped students develop their mathematical knowledge and problem-solving competence.

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GROUP–3

Examining problem posing

SEEKING AND USING STRUCTURE THROUGH PROBLEM POSING

Ioannis Papadopoulos, Nafsika Patsiala
Aristotle University of Thessaloniki

Successful problem posing demands an ability to reflect on the structure of the task. To achieve this the students' development of the habit of mind 'Seeking and Using structure' may be helpful. However, these two areas (problem posing and habits of mind) have not been connected yet. The pilot study presented here follows Grade-5 primary school students during a problem-posing intervention, we attempt to investigate how the accumulated experience on problem posing affects the development of this habit of mind. The findings give evidence that over time the students use more powerful problem-posing strategies that are relied on deployment of the problem's structure, and which is indicative of the development of the 'Seeking and Using Structure' habit of mind.

INTRODUCTION

Although an important topic, problem posing is much less common in mathematics classrooms than problem solving and thus far received less attention from the research community (Lester & Cai, 2016). In the existing research literature, several definitions of problem posing can be found (Papadopoulos et al., 2022). In this paper we adopt Silver's (1994) definition according to which problem posing is understood as "both the generation of new problems and the re-formulation of given problems" (p. 19). Problem posing is cognitively demanding and "often requires the poser to go beyond problem-solving procedures to reflect on the larger structure and goal of the task" (Cai et al., 2013, p. 60). The importance of structure in problem posing has been noticed early by Stoyanova and Ellerton (1996) who differentiate problem-posing situations according to their structure in three different types: free, semi-structured and structured situations. In the last two cases the students explore and reflect upon the structure of the given situation in order to pose new problems. Goldenberg et al. (2015) talk about mathematical habits of mind (HoM) as something that reflects how mathematicians think and describe them as "a way of thinking – almost a way of seeing a particular situation – that comes so readily to mind that one does not have to rummage in the mental toolbox to find it" (p. 3). Amongst others they refer to one called *Seeking and Using Structure* HoM as a readiness to seek and articulate underlying structure that might relate new problems to ones that have already been solved. Silver (2013) suggest that "other possibilities are also worth exploring, such as measuring the development of certain mathematical dispositions or habits of mind" (p. 161). In this spirit, a possible measurement of the habit of *Seeking and Using Structure* HoM could help to

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check the hypothesis that the accumulated experience on problem posing contributes positively to the development of it. Yet, these two areas of mathematics education (e.g., problem posing and certain habits of mind) have not been connected to our knowledge. In this pilot study, the impact of a problem-posing teaching intervention on the development of the students' HoM to seek and use structure is examined.

THEORETICAL BACKGROUND

The connection between problem solving/problem posing and understanding of the problem's structure has been highlighted by several researchers (Bernardo, 2001; Mamona-Downs & Downs, 2005; Schoenfeld & Herrmann, 1982). The term (mathematical) "structure" refers here to the underlying mathematical relationships between the entities and quantities within the given problem. Bernardo (2001) talks about *analogical problem construction*, a particular instructional strategy that allows students to construct their own analogous problems by encouraging them to explore the underlying structure of the given problem to identify its elements that are relevant to its solution. In this case, a successful construction of an analogous problem suggests an improved grasp of the problem structure achieved by the student. Generally, when given a problem-posing situation, the posers (students and/or teachers) are invited to explore its mathematical structure, and then they must use their knowledge, skills, concepts, and relationships from their previous mathematical experiences, to create one or more new mathematical problems (Baumanns & Rott, 2018). Kwek (2015) highlights the significance of the students' ability to identify the mathematical structure of the given problem as one (among others) cognitive factor influencing the learning and thinking that takes place during classroom-based problem posing. English conducted some problem-posing training programs for Grade 5 (1997a) and Grade 7 (1997b) students. A major component of the framework used in these programs was the students' ability to recognize and utilize the problems' structure. Her findings indicate that the effective development of problem-posing abilities demands special attention to this component. Indeed, the participating students were able to pose more diverse and more complex problems when recognizing the structure of the given problems. Similarly, English and Watson (2015) examined the potential of problem posing in developing the children's statistical literacy when working with primary school students. They claim that the significance of seeing and using the problems' structure was clearly seen in the students' responses.

Sometimes, the significance of the ability to see and use structure becomes evident in situations where the participants lack this skill. Prabhu and Czarnocha (2015) asked high school students to apply the rules of exponents to a given problem and then to make up their own problems using combinations of these rules. When not being able to observe the structure of the problem and the similarity of the structure with one or more rules, they had much difficulty in determining which rule was applicable for the problem under consideration.

Stoyanova (2005) classified the mathematics questions created by Grades 8 and 9 students while responding to a problem-posing prompt based on a specific question presented. The categories of strategies, such as reformulation and imitation, emerged. The core idea of these strategies relies on the notion of structure. Reformulation refers to a rearrangement of the elements in the problem structure in a way that preserves the nature of the problem. In the imitation strategy "the problem-posing product is obtained from the given problem-posing prompt by the addition of a structure which is relevant to the problem, and the problem-posing product resembles a *previously encountered or solved* problem" (p. 10).

The range of the studies that highlight the vital role of the use of structure in problem posing also includes teachers as participants. Sometimes these studies examine the results of an intervention. For example, Crespo and Sinclair (2008) in their study, asked prospective teachers to pose problems before and after an intervention. The results provided some evidence that the participating teachers became more sensitive on how the features and structure of the problems were amenable to modifications becoming thus more or less challenging for the students. Others are interested in the way preservice teachers approach problem posing given that there was not any instruction. In such a case, Chapman (2012) found that many of the participants exhibited an ability to pose interesting problems, but these problems did not make use of intentional or conscious consideration of the mathematical structure of the given problems.

In this landscape, the significance of the *Seeking and Using Structure* HoM for problem posing becomes evident, since it helps the students to see the logic and coherence in every new situation they encounter (Goldenberg et al., 2015). Aiming to develop an instrument to capture the students' ability to seek and use structure HoM, Patsiala and Papadopoulos (2022) considered the existing literature to find all the suggested problem-posing strategies. The successful application of these strategies is indicative of an awareness of the structure of the given problem. Afterwards, they approached specific experts in the area asking them to sort these strategies (Table 1) in three groups: A, B, and C and justify their sorting. The synthesis of their responds resulted in the final order of the strategies (from A to C) which is indicative of how powerful the strategy from the mathematical point of view is (from the most to the less powerful). The fourth category (D) was added in case the students' effort does not provide any evidence of a certain strategy. The instrument will be used to reliably record the progressive shift from one category to another over a longer period. The progressive shift of the students' choices from C to B and A type of strategies is considered indicative of the development of the *Seeking and Using Structure* HoM. A brief presentation of each strategy is given in the following (for more detail see Patsiala and Papadopoulos (2022).

Strategy	Category
The answer is a method	A
What-if-not	A
What-if-yes	A
Change the context	A
“Frontless” problems	B
Missing middle problems	B
“Tailless” word problems	B
Change the question/Form a question	B
Change numbers	B
Reversing known and unknown information	C
Change numbers	C
No evidence	D

Table 1. Problem posing strategies

In the “answer is a method” strategy the original question is kept but some numbers are left off and the required answer is a description of how the problem would be solved if the numbers were known. This strategy is connected to the notion of generalization, and it forces attention to the actual structure of a problem and what information is changeable.

The “what-if-not” strategy is the most known problem-posing strategy introduced by Brown & Walter (1983). Using this strategy refers to listing some attributes of the given problem and then to ask “What if each attribute was not so? What could it be then?”. The answer to this is the basis for posing the new problem.

On the antipode, the “what-if-yes” strategy (Leikin & Grossman, 2013) adds new properties to the given problem objects or adds new information which has an impact on the problem’s solution instead of challenging the problem’s conditions.

“Change the context” strategy refers to modifying the task environment (Tao (2006) calls it “aggressive” type of strategy) or changing completely the task environment (Mamona-Downs & Papadopoulos, 2017). Both of them indicate a deep structural understanding. In the latter the core idea remains the same, but it is transferred to another setting. As Ellerton (2013) stated, the process of designing a different context for a problem that has a similar structure to the given problem proved to be challenging for most students.

In "Frontless" problems only the question of the problem is kept. Given that the question cannot be answered without information, a potential solver is asked to add the missing data to complete the problem. It strongly resembles the questions we get in real life. As Goldenberg et al. (2015) say "In real life, the question poses itself first, and we must then figure out what information we need and what method we must use to answer it" (p. 18).

In "Missing middle problems" the original question is again kept but some numbers that are necessary for solving the problem are left out. The goal is to find which information is missing for the solution of the problem. In this case the answer might be again a number, but this number is part of the necessary information to solve the problem and not the answer to the problem's question.

In "Tailless" word problems the question at the tail end of the problem is omitted. The poser asks "what can you ask?" and a potential solver has to come up with a question to complete the problem. Obviously, the students might come with a question aimed at the same (presumed) goal of the original problem, but it is certain that there are other possibilities also.

The "Change the question/Form a question" refers to changing the existing question or posing from scratch a question that fits the given data. These two kinds of change are connected to different mathematical thinking. For most of the cases change in numerical data keeps the solvers close to the solution of the given problem whereas changing the question drives them to completely different situations from the given problem (Lavy & Bershadsky, 2003).

It is important to add here that although the "form a question" and the "tailless problems" strategy seem to have many similarities, there is an important difference between them. In the "tailless problems" strategy, the poser intentionally leaves the problem without question inviting the potential solver to find a question that can be answered according to the given information. In the "form a question" strategy, it is the problem poser who forms the question and then the potential solver is invited to solve a complete problem.

The strategy "Change numbers" refers to the change of some or all the numerical data of the given problem. In Table 1 this strategy appears in both B and C category because according to the kind of change (e.g., from whole numbers to fractions, from small number to big ones, etc) the result might be a simple or more complex situation as it will be demonstrated in the Results section.

Finally, in "Reversing known and unknown information" the given and the goal information of the initial problem are interchanged. Both of them are attributes of the given problem and students can deliberately change them to create a slightly different problem (Mose, et al., 1990; Whitin, 2006).

Given that Patsiala and Papadopoulos (2022) recently developed a tool for identifying the presence of this HoM based on the strategies used by the students, the research question of this study becomes: What is the impact of the accumulated experience on problem posing on the students' development of the Seeking and Using Structure HoM?

THE SETTING OF THE STUDY

The present case study is a teaching intervention and is the third in a series of small-scale pilot studies aiming to provide a deeper understanding of how experience on problem posing may have an impact on the development of certain HoM. Twenty-four Grade 5 students from a private school (convenience sampling) in Greece participated in this study. In this paper, we focus on the work of three students who provided us with an almost complete set of instances of the use of the problem-posing strategies. The intervention took place in parallel to the normal teaching of mathematics (i.e., students voluntarily attend a math club) and they had no prior experience on relevant problem-posing activities. The intervention lasted three months including twelve sessions of 45 minutes each.

The design of the intervention was influenced by two problem-posing frameworks: the Active Learning Framework (ALF) of Ellerton (2013) and the IMSTRA Framework (Immersion, Structuring, Applying) of Singer and Moscovici (2008). Firstly, the aim is to move students from being passive receivers to active learners through incorporating problem posing into instruction involving problem solving. A lesson based on this framework starts with the teacher inviting students to engage in the solving of model problems relevant to the new topic. Then, the teacher draws attention to similar problems in textbooks and the students solve them like the model problem. Then the students have to formulate their own problems that would have the same mathematical structure with the model one. Finally, class discusses and solves problems posed by students. The IMSTRA model includes three phases: Immersion, Structuring, and Applying with two sub-phases for each one of them. During Immersion the students "get immersed into the problem" by addressing and using previous knowledge, seeking information, planning, and performing experiments. Its first subphase (*Anticipation*) concerns the teacher helping students to formulate learning targets, identifying the knowledge necessary for the specific problem. During the second subphase (*Problem construction*) the teacher encourages students' explorations and helps them to record data. In the Structuring phase (with its two subphases *Synthesizing* and *Explaining*) students interpret their result from the previous phase and create new situations to check their claims. In the last phase (*Applying*) students apply what they learned to new situations by either solving existing problems or creating new ones that need solving.

Our intervention consisted of iterating cycles combining elements of both frameworks as described below. Each cycle consists of four phases (Figure 1). It starts with a problem-solving activity (PSA) (Phase 1) playing the role of model problem that is solved and discussed in the classroom. This phase reflected both frameworks. Afterwards, new problems are generated by the whole class (whole Class Problem Posing- CPP) inspired by the solved problem (Phase 2). This posing session is included in both frameworks but at the individual rather than on the classroom level. Typically, here the students must see the structure of the problem and apply various problem-posing strategies that fit the given data in order to be successful. The students can discuss with each other and put forward their ideas. They are encouraged to experiment and try to think differently than the problems they usually encounter in their textbooks. The produced problems are written on the board. Then (Phase 3), some of the posed problems are selected to be further discussed in class in terms of the strategy guided their formulation and of possible extensions of these problems (further Discussion of Posed Problems – DPP). They give the chance to recognize and highlight new problem-posing strategies. In the last phase of the cycle (Phase 4), a problem is chosen (from those generated by the students during Phase 2) and the students are invited to individually pose problems based on the given (Individual Problem Posing – IPP). This whole cycle is continuously repeated and essentially forms a spiral as each new cycle brings together their experience and potentially new strategies. In most of the cycles (except the first one), the model problem for the first phase was chosen from those created by the students in the last phase of the previous cycle. This gives students a sense of ownership of the problem (Kilpatrick, 1987) which results in a high level of engagement towards the whole process (Lavy & Shriki, 2010).

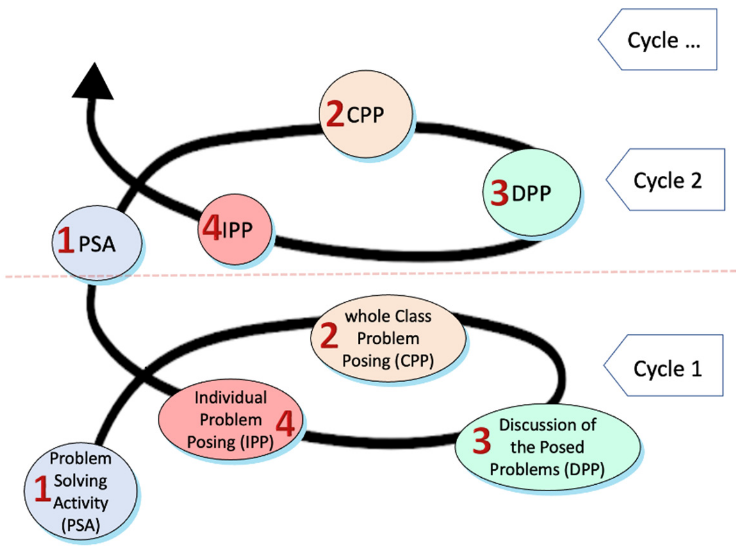

Figure 1. Problem-posing intervention cycle

It was during Phases 2 and 3 the different strategies could be extracted from the students' ideas and efforts to create new problems. This gave us the chance to highlight these strategies and discuss in the classroom the main principles of each strategy. For example, during one of the cycles the model problem was a trivial one: "I have three 10c coins and four 5c coins. What is the total amount of money

I have?". The problem was solved easily and during their collaborative effort to pose new problems, some students asked whether the total amount of money must be always the goal. They found it difficult to create many different problems with the same object being always the unknown, so they were wondering whether they are allowed to know the total amount of money and make the number of coins unknown. This initiated the discussion about the 'Reversing known and unknown' strategy.

In this pilot study, three problem-posing cycles took place. The problems given to the students at the last individual phase (IPP) of each cycle were respectively:

Problem 1: In my money box, I have 26 coins of two kinds and 10€ in total. What kind of coins and how many of them do I have?

This is a structured situation with a unique solution (6 coins of 1€ and 20 coins of 20c). In a sense it could be considered a multiple solution problem if we accept the involvement of zero coins as valid option. Then one can find several combinations of coins to get the total amount (20 coins of 50c plus 0 coins of any other kind of coins). The main elements of the problem are the total amount, the number of the coins, and the number of kinds of coins. The solution depends on the correct choice of the kinds of coins made by the solver.

Problem 2: I have 5 coins of two kinds, and I want to buy a book that costs 8.5€. Can I buy it?

This is also a structured situation involving an open problem having multiple solutions according to the kinds of coins involved. These solutions are based on the combinations of coins that result in amount of money equal or more than 8.5€ since to buy a book that costs 8.5€ it is not necessary to have the exact amount of money. So, any solution that leads to an amount bigger than 8.5€ is acceptable.

Problem 3: Every morning, Konstantinos and Lucia walk from home (A) to school (B). See the picture below and pose your own problems.

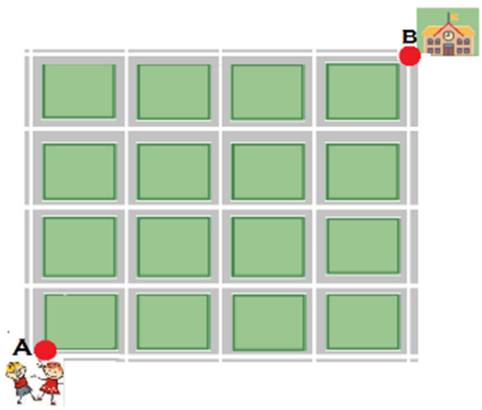

Figure 2. Task given during the 3rd cycle

Such a problem represents a semi-structured situation and lies in the realm of combinatorics. The most probable question would be to find the number of all possible routes from A to B. Translated in mathematics language the problem could be: Suppose we have a $n \times n$ grid with vertices placed as on the Cartesian plane. An object being at $(a, b)$is allowed to move to the point $(a+1, b)$ or $(a, b+1)$, which means we are always going left to right or bottom up. How many different routes from $(0,0)$ to (n, n)? The answer is the value of the binomial coefficient $\binom{2n}{n} = \frac{(2n)!}{n! \times (2n-n)!} = \frac{(2n)!}{n! \times n!}$. Another probable question might be to find the longest route or the shortest. This is a tricky question since all the routes have the same length which is n places to the right and n places up, $2n$ places in total.

The students were asked to pose problems that could be intriguing for them and their classmates. The individually produced problems at this phase constitute our data. These data were collected and categorized according to whether the formulated problems were indicative of the use of certain problem-posing strategies based on the collection of strategies presented in Table 1. This served as a way to monitor the efforts done by each student to examine whether more advanced problem-posing strategies were gradually used over time. All the responses were evaluated independently by the two authors in terms of strategies implied by the students' posed problems, and validity and reliability were established by comparing sets of independent results, clarifying categories until reaching consensus.

RESULTS

In total, 48 problems were generated by these three specific students: 13 problems were produced during the first cycle (1, 1, and 11 for the A, B, and C categories respectively), 16 during the second cycle (2, 6, and 8 or the A, B, and C categories respectively), and 19 (10, 4, and 5 or the A, B, and C categories respectively) during the third. Some representative examples from each category are presented in this section. Since category D concerns cases that show no evidence, such examples are not included in the results.

Category C problems

In the first two tasks, most of the students stuck with what they knew and either changed numbers imitating the problems they usually meet in their textbooks or reversed the known with the unknown information of the given problem after the question and the subsequent discussion made in the classroom. This is something expected as they are yet unaware of certain ways to pose new problems.

Therefore, in the first problem, the total number of coins, the total amount of money, and the number of kinds of coins were given while the goal was the number of coins of each kind. After employing the "Reverse known and unknown information" strategy, the produced problem was: "In my money box, I have 20c and 50c coins, in total 10€. What is the total number of coins I own?". So, in the new problem, the kinds of coins and the total amount of money are known while the total number of coins became the goal. The new problem is an open problem since there can be multiple combinations of 20c and 50c coins to get 10€ (e.g., 5 coins of 20c and 18 of 50c, or 10 coins of 20c and 16 of 50c to mention two of them). Similarly, for the second problem, the price of the book was considered known, whereas the number of coins became the unknown: "I purchased a book that costs 8.5€ given that I had in my pocket coins of two kinds. What is the possible number of coins in my pocket?" Again, the new problem is quite rich in terms of its multiple solutions.

An example of employing the "Change numbers" strategy can be seen in the next problem produced from Problem 2 given at the end of Cycle 2: "I have 10 coins of two kinds, and I want to buy a book that costs 7.50€. Can I buy it?" Here, the problem is the same as the initial one with slightly different numbers (e.g., 10 coins instead of 5, and 7.5€ instead of 8.5€). The new problem has the same complexity and structure as the original one and thus its solution is kept close to the solution of the given problem.

Category B problems

As mentioned earlier, the use of the "Change numbers" strategy results in problems that sometimes fit in category C and others in B. One of the students posed the next problem: "In my money box, I have 15 coins of four kinds and 20€ in total. What kind of coins do I have?". The student changed the numbers in Problem 1. However, the change made in the number of kinds of coins (four instead of two) adds extreme complexity to the problem since the possible combinations of coins increase. Thus, this time the specific strategy was considered as category B instead of C. Similarly, the next example is drawn from the next cycle: "I have 11 coins of three kinds, and I want to buy a book that costs 10€. Can I buy it?". Again, the increase in the number of kinds of coins results in much more successful combinations of coins to satisfy the conditions of the problem.

Another example of a category B strategy that was found during the third cycle is the "Form a question", which concerns posing from scratch a question that fits the given data. In Problem 3, the students must form questions based on the information embodied in the problem's representation (Figure 2). Each question must use the underlying structure. Two of the three students asked: "Find all the possible routes from home to school". This may seem a simple question, but it requires considerable effort to understand how the structure of the task situation leads to the pattern 'n-steps right and n-steps up', a discovery rather difficult for so young students.

Category A problems

The problem-posing strategies employed in this category are indicative of a deeper understanding of the problem's structure. Starting from the first problem that included the different kinds of coins and their total value, the new problem became: “A shepherd had goats and sheep. Each goat needs 2 kg grass per day and each sheep 3 kg per day. The shepherd needs 40 kg of grass every day. How many sheep and goats does he have?”

This is a nice example of applying the “Change the context” strategy. The setting in the initial problem is based on money, coins of different values, combinations of them, and a multiplicative relationship between the number of coins and the total amount of money. In the context of money, the main principle is exchanging ‘one of these for ten of these’ (one 10c coin for ten 1c coins, 1€ for ten 10c coins). The student kept the core idea but transferred it to another setting. Now, there are different animals, food of different weights, combinations of them, and a multiplicative relationship between the number of animals and the total weight of food. In the context of the weight the exchange is ‘one of these for one thousand of these’ (1kg for 1000g). The missing information in comparison to the initial problem was the number of sheep and goats together. Now, a new Diophantine equation is formed ($2x + 3y = 40$) and the problem is still solvable having multiple solutions (2 goats and 12 sheep, 5 goats and 10 sheep, etc.).

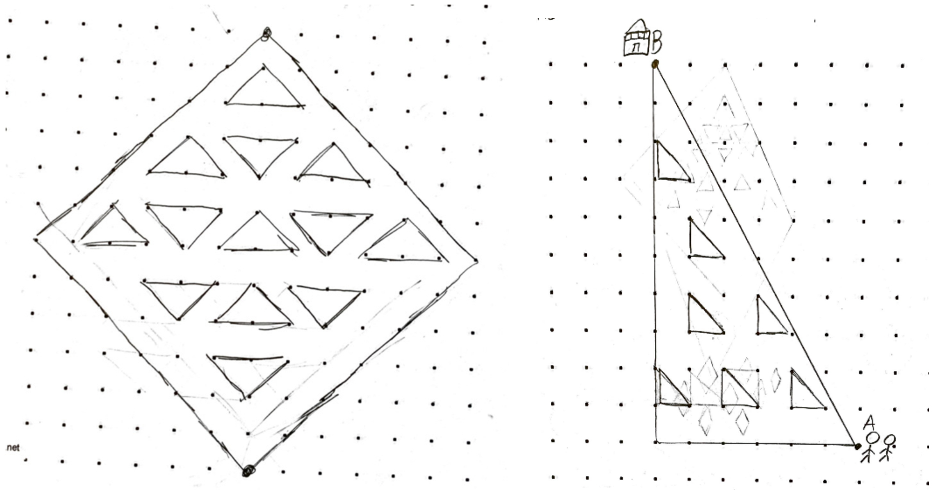

Figure 3. “What-if-not” strategy examples

In the third problem, some students decided to add more information employing thus the “What-if-yes” strategy: ‘What if the school is 3 km away and it takes them 15% of an hour to walk each km? How long does it take them to go to school?”. The length of the route and the amount of time per kilometer is a new information that has been added and is necessary for the problem’s solution. In the same spirit, others employed the “What-if-not” strategy. The students decided to challenge one of the attributes of the problem. In our case, it was the attribute of the shape in Problem 3: “What if the blocks were not rectangular but they were triangular and the outer shape was not a rectangle but a rhombus?” So, the student used a dot paper to draw it (Figure 3, left). There was also a suggestion of another student, in the same spirit, who wanted to use right triangles (Figure 3, right).

Obviously, both shapes do not provide a suitable structure for calculations compared to the given one in Figure 1. However, the students identified the initial structure and used this idea to create their own grid and using suitable shapes such as equal isosceles triangles (Figure 3, left) and equal right-isosceles triangles (Figure 3, right). The isosceles triangles in pairs form a rhombus (actually a square) similar to the big one. It seems that the student defined it as a rhombus because of its positions and not of its properties.

DISCUSSION

This research aimed to investigate the impact of a problem-posing intervention on the students' development of the *Seeking and Using Structure* HoM.

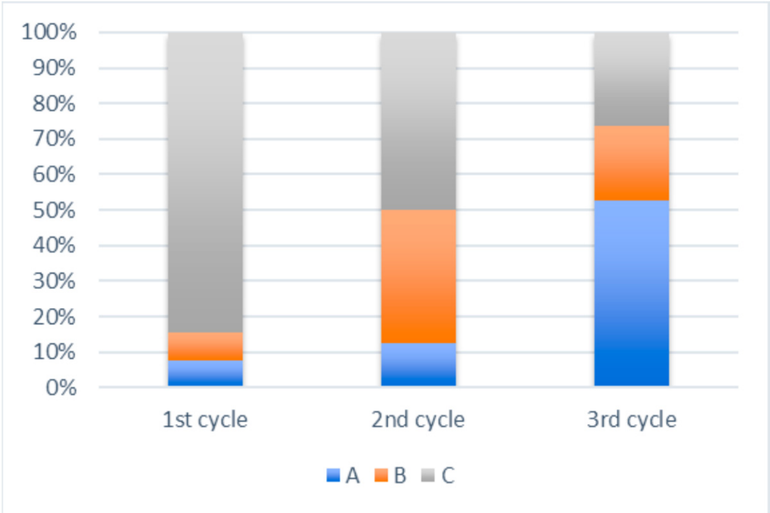

Figure 4. Distribution of the problem-posing strategies across categories and cycles.

The whole distribution of the produced problems per category of problem-posing strategy (A, B, and C) and cycle (1st, 2nd, and 3rd) is presented in Figure 4 which is indicative of a rather progressive shift in the students' employment of problem-posing strategies from category C to category B and even more to A. As it can be seen in Figure 4, the first cycle ends with the dominance of category-A strategies. This dominance is reduced as we move to the second and third cycles giving space to more powerful strategies (categories B and A).

At the end of the first problem-posing cycle most of the problems posed by the students employed mainly the "reverse known and unknown information" and "change numbers" strategies (category C). The students either simply mimicked the given problem by using the "change numbers" strategy or in their effort to produce different problems they preferred to make the given goal and the goal given. This is reasonable given the lack of knowledge on how new problems can be generated and it is considered as "surface reformulation technique" (Grundmeier, 2015) since it does not require the problem poser to change the structure of the problem but to change just a surface characteristic. As the discussions on the students' problems and strategies were progressively enriched during cycle 2, a considerable increase took place in problems that employed category B strategies as well as a slight increase in category-A strategies. In these cases, the students turned towards the "change numbers" strategy but in a more advanced level that led to more open and demanding problems yet still not requiring significantly in-depth inquiry into the mathematical structure of the existing problem. They also employed the "change question/form a question" strategy (also called modification strategy by Xie and Masingila (2017)). To form new questions means that the students must focus on the structure of the problem to see what they can figure out from the existing information. We were not able to identify the employment of more category B strategies at this step. Perhaps this is related to the short duration of the intervention that did not give opportunities for more strategies to emerge in the whole classroom sessions of problem-posing. In the third problem-posing cycle, significantly more cases of category A strategies were identified. The form of the task environment allowed students to notice that more than one sensible question could be asked. Students used both the "what-if-yes" and "what-if-not" strategies. Such strategies give students a framework within which they can think about how details contribute to what is being thought about and they give enough support that students continue to think rather than jumping to calculate, just "doing" something, or giving up. When done correctly during whole classroom sessions, and with chances for children to do it on their own, it can help children see that problems *do* have attributes and *can* be changed. This is indicative of a rather deeper understanding of the notion of structure.

One could object that if the structure of the problem in each next cycle is "easier to access" than the structure of the problem in the previous, then the progress from the first to the second cycle might be due to that easier-to-access structure. However, these strategies could be employed and emerge during every cycle no matter how easy the problem is. Moreover, the sequence of the chosen problems at the end of each cycle were chosen deliberately to progressively demand more thinking for accessing their structure. For example, in the first problem the total amount is fixed whereas in the second the wording of the task leaves open the option for having more than 8.5€ which makes the translation of the problem as well as the range of its solutions more demanding. Then, the third problem does not give any specific information and provides the students with the freedom and flexibility to navigate within the existing information to shape an understanding of the situation in order to decide what questions can be answered on the basis of this understanding.

CONCLUDING REMARKS

This paper describes a small-scale problem-posing intervention in primary school aiming to examine the impact of this intervention to the students' development of the Habit of Mind called *Seeking and Using Structure*. The findings give evidence that during this three-month interval there was a gradual shift in the students' choices from less powerful problem-posing strategies to more powerful ones, even though the former strategies rather dominate across the cycles. This is reasonable since they are the easiest to come to mind. However, this progressive shift towards more powerful strategies that are relied upon the full deployment of the problem's structure such as the "What-if-not" strategy can be considered a strong indication of the development of the *Seeking and Using Structure* HoM.

We acknowledge that this is a pilot study, the size of the sample was rather small, and the intervention was not long-term. Moreover, neither each strategy was equally chosen by the students, nor all the strategies emerged in the students' problems. Therefore, our findings cannot be overgeneralized. However, the evidence is strongly encouraging about the contribution of problem posing experience on the development of the *Seeking and Using Structure* HoM. Therefore, we aim soon to implement this intervention for a whole school year with more participating primary school students hoping that the longer the intervention, the deeper the development of this HoM.

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FOSTERING MATHEMATICAL CREATIVITY THROUGH PROBLEM POSING: THE USE OF "WHAT-IF…" STRATEGY

Marianthi Zioga[1] and Despina Desli[2]

[1]University of Western Macedonia, [2]Aristotle University of Thessaloniki, Greece

The present study explores two in-service elementary school teachers' perceptions and classroom practices regarding using the "What-if…" strategy to promote students' mathematical creativity. The participants' perceptions and teaching practices were examined through interviews and classroom observation before and after participating in an 18-hour educational program regarding mathematical creativity. Before the program, the participants did not use the "What-if..." strategy in their teaching. After the program, they both implemented the "What-if…" strategy when posing tasks, offering their students opportunities for creativity development. Findings highlight the need to increase teachers' awareness of mathematical creativity and problem posing strategies.

INTRODUCTION

Mathematics curricula from all over the world emphasize the need for teachers to propose mathematics investigations in their classrooms (Da Ponte, 2007). In order to do so, teachers are often required to modify the existing tasks found in mathematics school books. However, according to Leikin and Grossman (2013), the process of problem transformation is not simple; it requires specific problem posing skills and a solid mathematical background on the part of the teacher. The analysis of the definitions of "problem posing" reveals that there is no apparent consensus among researchers concerning the interpretation of problem posing as a mathematical and pedagogical construct (Papadopoulos et al., 2022). However, many scholars regard problem transformation (or reformulation) as an instance of problem posing activity (e.g., Kovács, 2017). The present study also approaches the concept of problem posing from the perspective of reformulating already existing or given problems, according to the classification of Papadopoulos et al. (2022). More specifically, Grundmeier (2015) defines problem reformulation as "*The process of posing a problem related to a problem that is or was the focus of problem solving*" (p. 414). Numerous problem posing strategies can be encountered in the research literature, among which is the "What-if…" strategy. According to Silver (1997), the implementation of the "What-if…" strategy in the classroom, as well as the "interplay" (p. 76) between problem posing and problem solving, is at the core of mathematical creativity.

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THEORETICAL BACKGROUND

The "What-if…" strategy

The "What-if-not?" strategy, described thoroughly by Brown and Walter (1993), is the most well-known problem posing strategy. When applying it, teachers and students are encouraged to change the initial conditions or goals of a previously solved problem (usually by negating them) to create new problems or pose new questions. In other words, they "challenge the given" by changing the specific content of a given problem. Pehkonen (1999) refers to this strategy as "problem variations" or "What-if" method. Furthermore, Leikin and Grossman (2013) propose the term "What-if-yes" strategy to describe dynamic changes that transform the initial problem by adding information or properties instead of changing them. In the present study, the term "What-if…" strategy is adopted to describe both "What-if-not" and "What-if-yes" strategies and includes problem variations and reformulation.

Research findings highlight numerous benefits regarding the implementation of the "What-if…" strategy in the mathematics classroom. For example, according to Brown and Walter (1993), teachers and students who apply it gain deeper insights into the givens and consequently strengthen their mathematical knowledge. In addition, Daher and Anabousy (2018) explain that teachers who implement the "What-if…" strategy not only move away from the stereotype that there is only one right way to approach a problem but also allow their students to discuss different ideas. Additionally, as Pehkonen (1999) describes, when the initial conditions or goals of a given problem are not taken for granted, students have the opportunity to work like creative mathematicians, posing questions and exploring different outcomes. Similarly, Silver (1997) endorses the belief that the "What-if…" strategy in the classroom can promote students' flexibility and creative disposition toward mathematics by allowing them to indulge in genuine mathematical activity, like professional mathematicians.

Mathematical Creativity

Creativity is usually evaluated based on the four indices proposed by Guilford (1966) and Torrance (1967). In the mathematical context, the four indices of creativity consist of fluency, flexibility, originality, and elaboration (Klavir & Hershkovitz, 2008). More specifically, fluency refers to the number of ideas a person has and the ability to produce many solutions to a given task. Flexibility is the ability to employ different strategies to solve a task or find solutions that belong to different categories and think in a non-algorithmic way. Originality refers to finding new, insightful, unexpected, and statistically infrequent solutions. Finally, as Lev-Zamir and Leikin (2011) explain, elaboration relates to a person's ability to incorporate detail into the solutions or make generalizations in the mathematical context.

In the past, mathematical creativity has been associated mainly with the work of prominent mathematicians. More recently, however, many researchers (e.g., Sriraman et al., 2011) perceive mathematical creativity as producing something new for oneself, even when the product is already known to others. In this sense, apart from prominent mathematicians, students can also be creative. However, although the importance of fostering creativity in educational settings is highlighted in mathematics curricula in many countries (e.g., NCTM, 2000), research findings reveal that teachers often find it challenging to recognize creativity and proceed with it in the classroom (Bolden et al., 2010; Desli & Zioga, 2015).

Educational programs regarding mathematical creativity and problem posing

The impact of teacher training on teachers' classroom practice regarding problem solving activities has always been of great concern. For example, when Kovacs and Konya (2019) organized a professional development program concerning problem solving for in-service teachers, participants' responses were very positive for the program, as most of them found it helpful. Furthermore, previous studies that examined the impact of educational programs regarding mathematical creativity (e.g., Levenson, 2015; Shriki, 2010) showed that the participants' perceptions of creativity after the program had been enriched, at least from a theoretical point of view. In Shriki's (2010) study, for example, before the program, the participants perceived mathematics as a "closed" domain where only mathematicians could be creative. In this context, the teachers are responsible for providing students with "final products" such as problems, rules, or concepts. After the program, however, they considered mathematics an "open" domain in which everyone could be creative. Moreover, they emphasized the creative process and, more specifically, the teachers' role in encouraging students to "generate" mathematics.

Grundmeier (2015) conducted an exploratory study that incorporated problem posing in a mathematics content course, aiming to develop prospective elementary and middle school teachers' problem posing abilities. More specifically, he focused both on problem reformulation and problem generation. Regarding problem reformulation, he found that the majority of the participants employed "*surface*" techniques (p. 422), which did not require the problem poser to change the structure of the problem. In other words, they changed only some of the surface features of the problems (e.g., numbers, the given, the wanted). "*Structure*" techniques (e.g., switching the given and wanted, changing the context), which are more cognitively demanding, were less preferred by the participants. However, as the educational course proceeded, the participants tended to rely less on surface techniques and more on structure and problem generation techniques, showing, according to Grundmeier (2015), that prospective elementary school teachers have the potential to pose complex mathematical problems.

The abovementioned studies have a specific limitation: participants' perceptions were evaluated through interviews, questionnaires, and writing assessments, but classroom observation was not incorporated. On the other hand, in a study by Lev-Zamir and Leikin (2013), which did not include an educational intervention, significant discrepancies were observed between participants' beliefs regarding mathematical creativity and their in-action practices. Consequently, as Levenson (2015) points out, although participants' theoretical perspectives regarding creativity and problem posing have evolved after the educational programs, we need to determine whether the skills acquired will be implemented in the classroom. To this end, classroom observation is crucial.

RESEARCH AIM AND QUESTIONS

The aim of the present study, which is part of a larger research project, is to explore elementary school teachers' perceptions and classroom practices regarding the use of the "What-if..." strategy as a means to promote the mathematical creativity of their students. Based on the abovementioned theoretical considerations and empirical results, the following research questions guided the study: a) What are the participants' perceptions and teaching practices regarding the use of the "What-if..." strategy?, b) What are their respective perceptions and practices after completing an educational program regarding mathematical creativity? and c) To what extent does the use of the "What-if…" strategy offer opportunities for fostering creativity in the classroom?

METHOD

Similar to other studies that aim to explore teachers' perceptions and the consistency between them and their teaching practices (e.g., Lev-Zamir & Leikin, 2013), the present study collects information using interviews and classroom observation. According to Bryman (2016), interviews are a valuable tool for collecting data about the participants' attitudes and beliefs. Although the data from interviews are not generalizable, they can offer a new perspective. Interviews' "*exploratory nature*" (Nathan et al., 2019, p. 391) allows the researchers to answer questions about which little is known. On the other hand, classroom observation offers the opportunity to examine the participants' teaching methods in their real-life environment. This way, discrepancies between their expressed beliefs and their practices can be identified.

In order to examine whether (and how) teachers' beliefs and practices evolve after participating in the educational program, following Levenson's (2015) example, data is gathered at different times: before and after their participation in the program. The program aimed to increase teachers' awareness of mathematical creativity so that they could implement creativity-fostering approaches in their classrooms.

Participants. Two Greek in-service elementary grade teachers, Peter and Helen, were the main participants. They were chosen among seven teachers who participated in the original research project since they were the only ones to implement the "What-if…" strategy during classroom observation. Peter has 15 years of teaching experience and holds a Master's degree in Mathematics and Science Education. Helen has eight years of teaching experience and holds a Master's Degree in Language Education. They both teach in Grade 4 and work in public elementary schools in the same city (Thessaloniki, Greece). Their participation in the study was voluntary and anonymous.

The educational program. The educational program was conducted by the first author of the study and lasted for a total of 18 hours, equally distributed into six three-hour sessions. The participants were presented with research findings and approaches related to mathematical creativity and the ways to promote it. The program's focus was not limited only to problem posing but incorporated a variety of approaches and tasks that cultivate mathematical creativity, with particular emphasis on the promotion of the four indices of creativity. The "What-if…" strategy was discussed, among others, using examples from the research literature (e.g., Brown & Walter, 1983; Hashimoto, 1997; Mihajlović & Dejić, 2015; Pehkonen, 1999; Silver, 1997). First, specific problems chosen from the abovementioned literature were presented, and their variations were discussed, along with delineating the cognitive benefits for the students. Afterwards, the participants were encouraged to modify and solve mathematical tasks using the "What-if…" strategy.

Instruments. Data was collected through semi-structured individual interviews as well as classroom observation. The open-ended interview questions allowed the participants to respond freely and express their beliefs openly. The interview aimed to gain a deeper understanding of the participants' perceptions regarding mathematical creativity: its nature, the ways it can be expressed and cultivated in the classroom, the tasks that offer opportunities for creativity development, and the role of the teacher to this end. The following were examples of the questions asked: "*What is mathematical creativity?", "Can it be fostered in the classroom?", "What can a teacher do to promote students' mathematical creativity?*". The questions deliberately did not refer directly to the "What-if…" strategy or any other teaching approach. Instead, they were more general, allowing the participants to express themselves freely and spontaneously.

The participants were aware that classroom observation aimed to examine their teaching practices regarding the promotion of mathematical creativity. For the purpose of the study, a structured observation sheet was created and used. Classroom observation focused on the participants' teaching approaches and choices of tasks that favoured mathematical creativity. More specifically, every time a participant implemented a creativity-promoting task, his/her methods, their frequency and duration, as well as the specific details of the task, were written down (e.g., use of the "What-if…" strategy, encouraging students to find many solutions to a task, encouraging students to think in a non-algorithmic way).

Procedure. The study was conducted in two phases, with participants being examined before and after attending the program. During the first phase, the participants' perceptions of mathematical creativity and tasks that foster it were explored through the interviews. The interviews lasted approximately 30 minutes for each participant and were recorded and transcribed. In addition, four hours of classroom observation during mathematics lessons took place for each participant. After the completion of the program, the second phase of the study took place. During the second phase, interviews with the participants and classroom observation were conducted again to explore changes in the participants' perceptions and teaching practices. Similar to the first phase, four hours of classroom observation took place for each participant.

RESULTS

The results from classroom observation showed that before the program, the participants did not use the "What-if…" strategy at all during their teaching. After the program, however, both participants implemented the "What-if..." strategy multiple times when posing problems.

Peter's perceptions and teaching practices before the program

In the interview before the program, Peter was asked, "*Which tasks, in your opinion, have the potential to promote students' mathematical creativity*?" and elaborated on the importance of the "What-if..." strategy. More specifically, he explained that he often extends or modifies a previously solved problem to enhance students' creativity, changing some of the givens and posing questions like "*What would happen if...*". He also commented that by using suitable strategies, almost every problem could potentially promote students' mathematical creativity.

> I try to choose as many (creativity-fostering tasks) as possible, either from the schoolbook or by "surprising" the students and extending a previously solved problem they already had the chance to process. (I extend it) by changing some of the givens or adding or removing some of the givens. For example, "If that was missing, what would we do?". I suppose that, in this sense, any problem can potentially promote creativity in the long term. We use this as a game in the classroom, so the students are much more motivated than they would be while solving a routine task.

However, although during the interview Peter mentioned the benefits of implementing the "What-if…" strategy, he did not use it during the four hours of classroom observation before the program. This fact reveals a certain inconsistency between his perceptions and his practices.

Peter's perceptions and teaching practices after the program

After the program, Peter mentions again that problem extension and modification are creativity-fostering strategies. Furthermore, he states that the modification of the givens can occur not only by the teacher but also by the students. He explains that to enhance students' mathematical creativity, teachers can encourage them to pose problems by reformulating or extending a previously solved problem or by adding some givens. Although he does not use the exact term, the process he describes relates to the "What-if-yes" strategy. He also mentions the possibility of finding more solutions (fluency) if a given is removed from the problem, describing (though not explicitly) the "What-if-not" strategy.

> I ask (the students) what will happen if we change a parameter, a given. For example, can we find the same or maybe more solutions if we remove this given? I also ask students to extend a problem further, for example, how would we use some additional or different givens and what solutions would we find.

During classroom observation after the program, Peter designed a problem, which he reformulated multiple times. More specifically, he created the following problem: "*Steve took a loan of 24.000€ and agreed to pay it back in monthly instalments. He could not pay more than 500€ per month. How many months will it take him to repay the loan?*". A student proposed solving it using the vertical division algorithm and found that 24.000 : 500 = 48. "*Correct*", said Peter, "*but, can someone solve it without using the algorithm?*".

Another student then proposed a different way of solving the problem, according to which "*If he paid 1.000€ per month, it would take him 24 months to repay the loan. Now that he pays 500€ per month, which is half, it will take him twice the time, which is 48 months*". Apparently, the student realized that the monthly instalment amount and the number of months required for the repayment are inversely proportional amounts, although he did not state it expressly. It is possible that he came to this solution using his logic and previous experience. First, he divided the whole amount of 24.000 by 1.000 (a convenient number). Then he used the property of inversely proportional amounts that if one variable decreases (the monthly instalment), the other (the number of months) increases in the same proportion.

It is interesting to note that Peter encourages his students to avoid using the algorithm of vertical division and promotes a more creative, non-algorithmic, flexible way of thinking. Peter then used the "What-if…" strategy to pose more problems. More specifically, he first asked: "*We know that he cannot pay more than 500€ per month. What if he pays less?*". "*Then it would take him more months*", replied a student, and all the others agreed. This way, Peter helped students clarify that the monthly instalment amount and the number of months required for the repayment are inversely proportional amounts. However, he did not state it per se, as students in Greece are taught about proportional and inversely proportional amounts later, in Grade 6.

He next asked: "*What if he pays 200€ per month? Can you solve it without division?*", to which a student replied: "*120 months because 200·120=24.000*". In other words, the student made use of the fact that multiplication and division are inverse operations. Peter continued to modify the givens of the problem: "*What if he pays 400€ per month?*". Another student answered: "*60 months, which is half than before because the amount is double*". The student employed the result found from the previous question and the properties of inversely proportional amounts. Peter reformulated the problem a few more times: "*What if he pays 300€ per month?", "80 months because 300·80=24.000". "What if he pays 100€ per month?", "240 months because 100·240=24.000". "What if he pays 250€ per month?", "We previously found that paying 500€ per month will take him 48 months. So, if he pays 250€ per month, which is half, it will take him twice the time, which is 96 months*".

The way Peter approaches the specific problem encourages the development of students' mathematical creativity. First of all, the initial problem posed by Peter is an ill-structured one: the parameter of the monthly instalment is not precisely defined, allowing for the exploration of many different solutions and the development of students' fluency. Peter also reformulates and modifies the initial problem multiple times, posing new problems using the "What-if…" strategy. By doing so, he encourages his students to explore many different occasions and look at the initial problem from many different perspectives, thus promoting their flexibility. Students' flexible way of thinking is developed by another approach as well. Peter encourages his students multiple times to perform convenient mental calculations instead of using the algorithm of vertical division. By doing so, he motivates his students to think in a non-algorithmic, non-stereotypical way and employ different strategies and properties to solve the problem.

It is evident that, after the program, Peter's practices regarding the use of the "What-if…" strategy are much more enriched and also consistent with his perceptions, as he expressed them in his interview.

Helen's perceptions and teaching practices before the program

The data collected before the program show that Helen's perceptions and teaching practices under the scope of mathematical creativity were minimal. Regarding the "What-if…" strategy, she did not mention it during the interview, nor did she use it at all in her teaching.

Helen's perceptions and teaching practices after the program

After completing the program, Helen did not mention the "What-if…" strategy among the creativity-fostering approaches in her interview. However, her classroom practices have been enriched. More specifically, she modified two tasks to encourage students' divergent thinking.

The initial problem Helen assigned to her students (which she subsequently modified) was one that had been discussed during the educational program:

«*Design shapes with an area of 24 sq cm*».

Students came up with three answers: the three rectangles shown in Figure 1. The particular problem promotes students' fluency and divergent thinking, as students are encouraged to find many solutions. It has the potential to promote originality as well if the students are encouraged to find unexpected solutions that no one else will (for example, a rectangle with a width of 1cm and length of 24cm or a width of 0.5cm and length of 48cm and many more). However, Helen did not encourage her students to find original solutions and missed that opportunity. Moreover, Helen could have encouraged her students' flexibility by asking them to design different shapes other than rectangles (for example, triangles). However, she missed that opportunity as well.

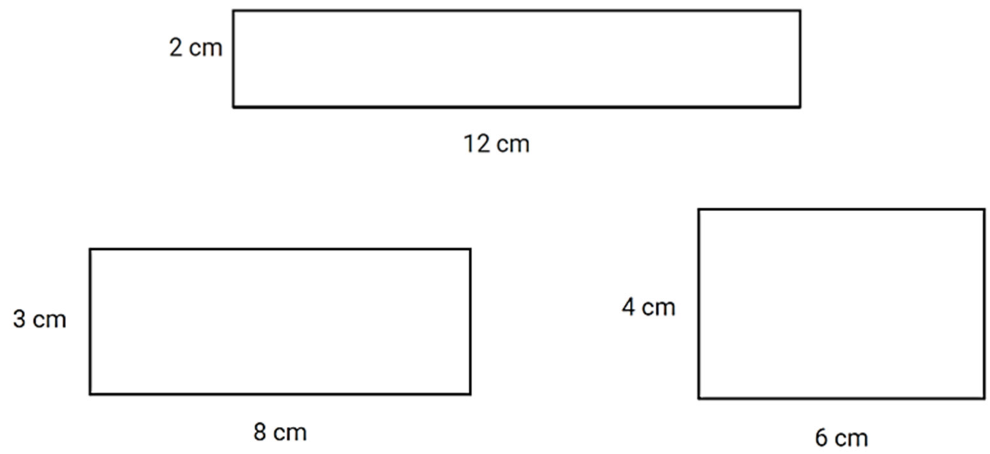

Figure 1. The answers proposed by Helen's students for the first problem

She next posed a different problem, using the "What-if…" strategy:

"*What if the area was 25 sq cm? What shape could it be?*".

At first, many students came up with the solution of a square with a side of 5cm.

"*Other than a square, what else could it be?*" asked Helen, and her students also discovered the two rectangles shown in Figure 2.

By changing the givens of the initial problem, Helen offered her students the opportunity to develop their fluency by finding many solutions to the problem and develop their flexibility and non-stereotypical way of thinking by looking for different shapes apart from the square.

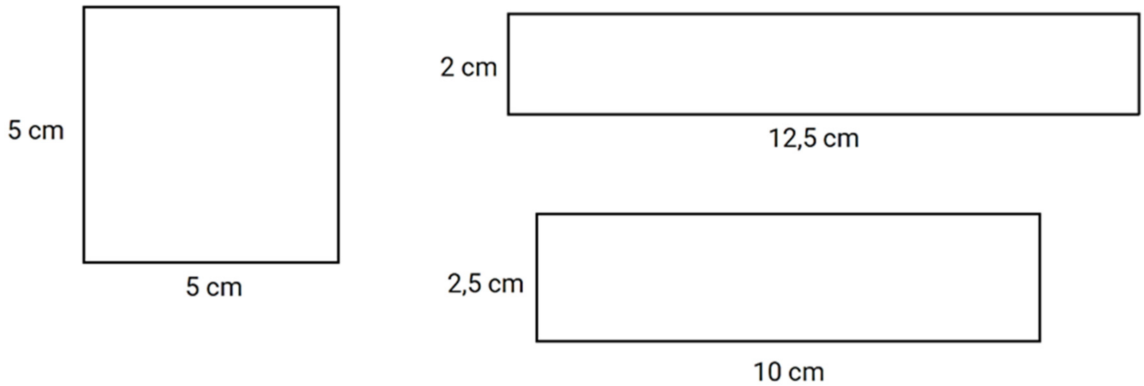

Figure 2. The answers proposed by Helen's students for the second problem

Then, she assigned a different task, which was taken from the school mathematics textbook:

"*Design a square decimeter. Then paint red an area equal to 1/10 of the whole area*".

Almost all of the students painted a column. "*What if we do not paint a column? What else can we paint?*" she asked, encouraging them to think more flexibly and less stereotypically. Students then proposed painting a row or the diagonal (see Figure 3). After the end of the lesson, Helen explained to the researcher that this variation was a spontaneous decision that was not predetermined during the lesson plan.

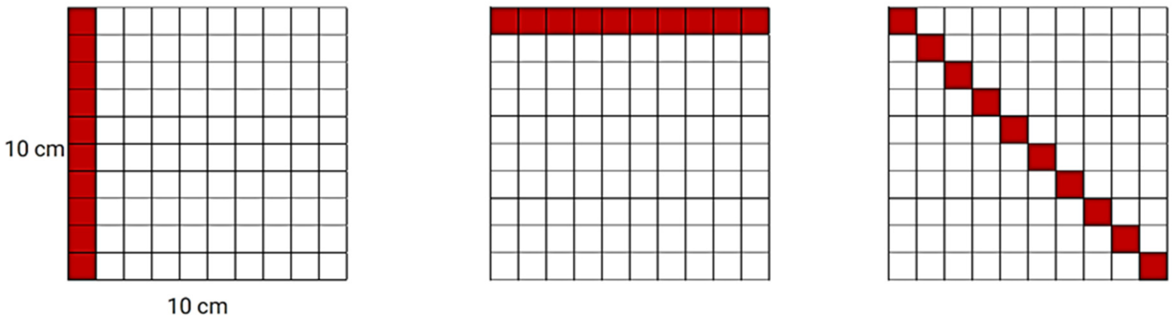

Figure 3. The answers proposed by Helen's students for the third task

Judging by Helen's approach, the "What-if…" strategy offers many opportunities for developing students' mathematical creativity. First, Helen promoted students' fluency, asking them to come up with many solutions and design many shapes with a given area. Also, she promoted students' flexibility to an extent, encouraging them to think of shapes other than the square (in the second task) or the column (in the third task). However, she could have also asked for shapes other than a rectangle. Furthermore, the employment of the "What-if…" strategy offered the potential to develop students' originality by encouraging them to find rare and unconventional solutions. On both occasions, however, she missed the opportunity to promote originality and encourage students to find shapes that no one else would.

Interestingly, Helen used the "What-if…" strategy and implemented it in two different situations during the four hours of classroom observation after the program, even though she did not mention it in her interview.

DISCUSSION

The present study attempted to investigate the perceptions and classroom practices of two in-service elementary school teachers regarding using the "What-if…" strategy, which constitutes a link between mathematical creativity and problem posing (Silver, 1997). Findings showed that both teachers responded positively to implementing the "What-if…" strategy when posing problems.

More specifically, regarding the first research question, participants did not use the "What-if…" strategy in their teaching before attending the educational program. Although Peter mentioned the benefits of the "What-if…" strategy as a means to promote mathematical creativity and explained that he often implements it, he did not actually employ the "What-if…" strategy in his teaching during classroom observation. This inconsistency is in accordance with Lev-Zamir and Leikin's (2013) findings; in-service teachers' perceptions and classroom practices regarding mathematical creativity were examined, and a "significant gap" (p. 306) was found between teachers' theoretical beliefs and classroom practices concerning the cultivation of mathematical creativity. However, from a different, more optimistic point of view, it is possible that in this case, the four hours of classroom observation were insufficient for his whole variety of teaching strategies to unfold.

Concerning the second research question, both participants' classroom practices regarding using the "What-if…" strategy to promote mathematical creativity evolved after participating in the program. Peter elaborated on the importance of problem reformulation and repeatedly modified a problem he had created. Helen, on the other hand, used the "What-if…" strategy twice during classroom observation, although she did not mention it in her interview. This inconsistency, which favours her teaching approaches, probably reveals that it is easier for Helen to change and improve her classroom practices than to change her theoretical background and beliefs. Such a shift will likely take more time and many more educational programs for teachers to attend.

Furthermore, similar to the participants in the early stages of Grundmeier's (2015) study, Peter and Helen utilized surface problem reformulation techniques. More specifically, they only changed the givens of the initial problems; they did not interfere with the structure of the problems (for example, they did not switch the given and wanted, change the context, or extend the original problem). Their choice of techniques could be explained due to their lack of experience in problem posing. As Grundmeier (2015) explains, structure reformulation techniques are cognitively demanding and require a deeper understanding of mathematical content or more problem posing experience on the part of the teacher. One should also take into consideration that Peter and Helen were the only participants (among seven) in the original study that implemented the "What-if…" strategy during four hours of classroom observation (for each participant). Hence, findings from classroom observation reinforce Grundmeier's (2015) conclusion that elementary school teachers do not employ structure reformulation techniques and need an environment rich in problem posing experiences during their undergraduate studies to be able to do so.

In regard to the third research question, findings reveal that after the program, both participants offered their students numerous opportunities for creativity development, by employing the "What-if…" strategy when posing problems. More specifically, both teachers encouraged their students to look for different solutions to given tasks and for less algorithmic or stereotypical solutions urging them to develop their fluency and flexibility. This finding is consistent with Levenson's (2015) results concerning a teacher who, after participating in a creativity educational program, endorsed the view that creativity is characterized by fluency and flexibility and by overcoming stereotypes. Her classroom practices, however, were not observed. Interestingly, although the problems assigned by Helen had the potential to promote originality as well, she did not take advantage of this opportunity. It seems that teachers find it relatively easier to pose problems that cultivate fluency and flexibility than tasks that cultivate originality or elaboration. Teacher educators should consider these difficulties and include problem reformulation in mathematics courses to address all aspects of creativity enhancement, emphasizing originality and elaboration, along with fluency and flexibility.

Apart from the implications on teachers' education, it would be interesting to search for this finding further; it seems that there is a need to examine teachers' difficulties in enhancing students' originality and elaboration. However, research findings regarding the development of students' elaboration are surprisingly limited since it seems difficult even for the researchers to assess it. More specifically, as Kozlowski et al. (2019) explain, although elaboration is the fourth index of mathematical creativity, current instruments used by researchers to evaluate mathematical creativity do not include elaboration in their data collection; there appears to be a need for better and more upgraded creativity assessment tests.

It is worth mentioning that Peter's knowledge of the "What-if…" strategy and its potential to promote creativity, even before participating in the program, can probably be justified due to his Master's degree in Mathematics and Science Education. His post-graduate education has potentially made him more aware of specific approaches in mathematics teaching than Helen, whose Master's degree is in Language Education. This potential relationship between teacher studies and the emergence of creativity in teaching formed the core of Lev-Zamir and Leikin's (2013) argument, who suggest that teachers with a stronger mathematical background feel more confident in choosing creative tasks and encouraging creative solutions.

The results of the present study should be interpreted with care, since the small number of participants does not allow for more conclusive results. Furthermore, the study examines teachers' classroom practices and does not evaluate the actual development of students' creativity. The participants encouraged their students to come up with different solutions or think more flexibly and search for different strategies, but students' fluency and flexibility were not assessed. This could be the focus of a future study. For example, students' mathematical creativity could be evaluated using a creativity assessment test before and after their teachers' participation in an educational program regarding creativity. This way, the program's impact on students' creativity development could be examined.

In summary, the study's findings highlight the benefits of implementing the "What-if…" strategy in the classroom, as it offers the potential to promote students' fluency and flexibility and, consequently, mathematical creativity. Findings also indicate the need to increase teachers' awareness of mathematical creativity and to incorporate problem posing experience in teacher education.

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EXPLORING STUDENTS' ALGEBRAIC THINKING THROUGH PROBLEM-POSING ACTIVITIES

Emőke Báró

University of Debrecen, Hungary

MTA-ELKH-ELTE Research Group in Mathematics Education, Hungary

Based on the literature problem-posing is one possible way to observe students' thinking. We (the author collaborating with two university experts) designed a chapter from the curriculum suitable for online learning that includes problem-solving and problem-posing. Sixty-one seventh-grade students were asked to solve patterning problems and pose problems based on model problems. This paper aims to analyze the characteristics of students' problem-posing concerning patterning activities.

INTRODUCTION

Mathematics has been considered a cultural artifact, something we receive as part of our cultural heritage (Kaput, 2008). This cultural artifact, particularly algebra, is embedded in education systems worldwide in various ways, especially regarding when to introduce algebra and how closely it is integrated with other mathematical topics (Kendal & Stacey, 2004). Being aware that lower secondary students are in a transitional phase from arithmetic to algebraic thinking, we wanted to examine their way of thinking. We were intrigued by the book of Rivera (2013), according to which, in the process of problem-solving, secondary students, even adults are more likely to provide an empirical (numerical or visual) than a formal explanation. As Rivera, we also argue that more patterning activities might provide an opportunity to improve formal explanations and deepen students' algebraic thinking.

Therefore, we (the author with two university experts) designed a teaching project transforming one chapter from the curriculum, including problem-solving and problem-posing. This teaching project was designed to be suitable for online learning and to observe the possible transition from arithmetic to algebraic thinking. The study provides an example of how to put into practice patterning activities and problem-posing in grade 7 to map students' way of thinking and explore their arithmetic or algebraic thinking processes.

We formulated our research question based on Rivera's (2013) findings. We wanted to determine whether 7^{th}-grade students reason more algebraically or arithmetically when posing and solving patterning problems. Specifically, how is

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algebraic thinking reflected in students' problem-posing activities after patterning experiences?

THEORETICAL BACKGROUND

In mathematics education, a problem implies an obstacle that hinders the achievement of a goal. The way to overcome the obstacle is problem-solving and purposeful reasoning (Polya, 1962). Heuristics have been generally recognized as a crucial component for problem-solving (Mayer, 2003) because they are general suggestions on a strategy that is designed to help when we solve problems (Schoenfeld, 1985). According to Tiong (2005), thirteen heuristics have been identified that can be applied to mathematical problem-solving at the lower secondary level, one of them being "looking for patterns." Patterning or pattern recognition is the search for regularities and structures (Clements & Sarama, 2009). Many different kinds of patterns in the school mathematics curriculum can be represented numerically or figurally (Rivera, 2013) and can be used in the classroom to promote problem-solving and posing.

Pehkonen (1997) regards problem-posing as a particular type of problem-solving, indicating that problem-posing and problem-solving are deeply connected. Moreover, Brown and Walter (2004) claim that problem-solving may result in problem-posing. As students are encouraged to raise questions and pose problems of their own, rather than to only solve them, their learning becomes more active. Recently, there has been an increased interest in problem-posing (Koichu, 2020), which has resulted in different definitions of problem-posing (Baumanns & Rott, 2021). Silver's (1994) traditional approach is well-known in literature as it includes both the invention of new problems based on particular situations and the reformulation of existing problems. Stoyanova and Ellerton (1996) define problem-posing as students creating personal interpretations of specific situations and transforming them into mathematical problems. In this paper, we use the definition of Cai and Hwang (2020, p.2) according to which problem-posing includes "several related types of activity that entail or support teachers and students formulating (or reformulating) and expressing a problem or task based on a particular context."

Problem-posing can be brought into the classroom in different ways. We aimed to combine problem-solving and problem-posing during an online lesson to observe these patterning activities, paying particular attention to students' way of thinking: is it rather arithmetic or algebraic? Numerous researchers have pointed out the relationship between algebra, patterns, and generalization. For example, Kaput (1999) defines algebra as "the generalization and formation of patterns and constraints" (p. 136). Usiskin (1988) mentions the first out of five different concepts for "algebra as generalized arithmetic" (p. 11). Most attempts to define algebra historically tend to be oriented toward solving equations, where the origin of the equations might be problem situations (Katz, 1995). Franke et al. (2008), suggest a notion of examining expressions and equations entirely, not as a step-by-step process, referring to it as relational thinking. Using a modern perspective (18^{th} century and later), definitions of algebra refer to the use of literal symbols as a central feature of the activity (Kaput, 2008).

Usually, schools start teaching mathematics with numbers and numerical procedures (arithmetic) and later proceed to symbols and symbolic procedures (algebra). The linkage between numbers, symbols, situations, and problems is often missing (Smith & Thompson, 2007). We regarded problems with patterns as a bridge between arithmetic and algebra; as students need more experience in observing patterns and making generalizations before using variables (Schoenfeld & Arcavi, 1988; Lócska & Kovács, 2022).

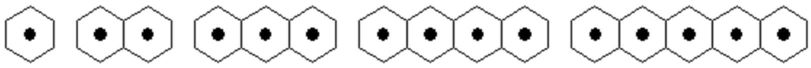
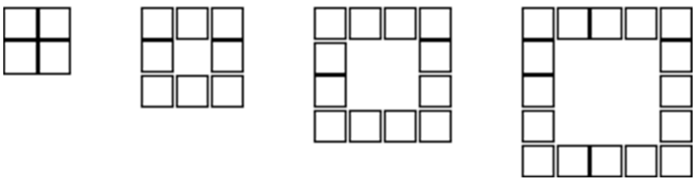
THE SETTING OF THE STUDY

Two seventh-grade classes took part in the study, a total of 61 students, 13-14 years old from Transylvania, Romania (Class A: 31 students, Class B: 30 students). Among them, there were two students with special needs. Due to the covid, some students only took part in online lessons but didn't send their solutions to the teacher. The instruction language was Hungarian since Hungarian was the maternal language of the students. Action research has been implemented. Action research happens when people are involved in researching their practice to improve it and come to a better understanding of their practice situations (Feldman, 2017). The action research reported here involves one mathematics teacher–researcher (the author) from Romania teaching 7th graders, and two university experts in mathematics education. We aimed to design a chapter according to the curriculum, taking into account previous teaching–research experiences. The title of this chapter was: Equations and problems that can be solved by equations of the form $ax + b = c, a \neq 0$. The students had previous knowledge about first-degree equations in one variable and the balance method from grade 6.

Data collection and analysis

To document the study, different research instruments were used. Due to the pandemic, all lessons were held online, using Google Meet and editable shared documents. The shared documents allowed the teacher to monitor the participating students' work in real-time. The lessons were recorded, which helped us in reflecting and analyzing student work. Every worksheet filled out by the students was photographed, their notebooks containing their homework were scanned, and every online document was saved. These documents helped us to interpret students' oral manifestations.

This paper analyzes students' problem-solving and problem-posing activities connected to the problems listed in the following table, see Table 1. The tasks are divided into different parts according to the teaching method used.

Task	Teaching method
Task 1 (Mason, 1988)*:* You are given the following pattern. Make a table showing the number of points and line segments in each figure. 	Frontal, teacher explaining
Find a rule between the number of points and line segments. Explain the rule!	Online work in pairs
The students present and compare different answers	Class discussion
Answer the following questions: a. How many line segments are in the figure with 7 points? b. How many line segments are in the figure with 12 points? c. How many points are in the figure with 46 line segments? d. How many points are in the figure with 50 line segments?	Individual task
The students present and compare different answers	Class discussion
Task 2 (Mason, 1988): The figures below are made of square tiles. Make a table showing the number of tiles in each figure. 	Frontal, teacher explaining
Answer the following questions: How many tiles will there be in the 9th figure? How many tiles will there be in the 20th figure? Which figure will have 98 tiles? Generalize: how many tiles will there be in the nth figure?	Online work in pairs
The students present and compare different answers	Class discussion
Task 3: Design a sequence of figures (you can use matches, toothpicks, ear picks) by specifying at least its first four elements, then ask two questions and answer them! Send in your sequence, questions and solutions!	Homework, problem-posing

The students present and compare different answers	Class discussion

Table 1. Tasks of problem-solving and problem-posing

Table 1 above shows that the work in pairs and the individual work was followed by classroom discussions to reduce possible errors. The pair work part was already discussed in a previous paper (Báró, 2022). Shortly, we found that students had managed to identify the rule(s) successfully, and we had sorted the reasoning ability into three categories: (1) the students explained the rule, and the reasoning is correct; (2) the students wanted to explain the rule, but they couldn't; (3) the students did not explain at all.

Problem-solving has been analyzed in each task. Since *Task 3* required problem-posing and solving, we analyzed the posed problems by creating a coding frame.

Coding students' work

We processed the students' documents through qualitative content analysis. As a first step, we developed a coding frame for evaluating students' work in two main dimensions: problem-posing and problem-solving. The coding frame was based on previous studies (Lócska & Kovács, 2022; Kovács et al., 2023) and our preliminary review of the students' work. Concerning the quality of the posed question, the following codes were developed (see Table 2). In each case, we examined whether each of the posed questions is relevant to the pattern and task.

Codes	Definition
Relevant (R)	At least one relevant question was asked. In the case of one relevant, and no irrelevant question, we regard the lack of a second question as student inattention.
Irrelevant(IR)	One or two irrelevant questions were asked, and no relevant question was asked – i.e., the student submitted no interpretable or mathematically meaningful question
0	The student posed a pattern, but no question was asked, OR one relevant and one irrelevant question were asked.
No pattern (NP)	The student did not define a pattern, e.g., only the first element of the pattern was given.

Table 2. Coding the quality of the posed question

Question types were coded, see Table 3, if the questions were formulated correctly, i.e., the student's work received the code R based on the previous criterion. We can distinguish two types of questions regarding the required way of thinking for the solution: (1) forward ($n \to a_n$) when the location of the figure is given and the number of the components needs to be found; or (2) backwards

($a_n \rightarrow n$) when the number of components is given and the location of the figure has to be found. In each case, we considered whether the question required the finding of a close or far, or even a generalized element.

$n \rightarrow a_n$	Close (C)	The student poses a question about not more than the seventh member of the sequence (close element).
	Far (F)	The student poses a question about at least the eighth member of the sequence (far element)
	General (G)	The student asks for generalization.
$a_n \rightarrow n$	Backwards (B)	The student poses a question whose solution requires backward thinking (e.g., Task 1. c, d)

Table 3. Coding the type of the posed questions

The quality and type of the solutions to the posed problems were also coded (see Table 4). In this case, we coded both the type of correct and incorrect solutions, because we were interested to identify the type of thinking that characterizes these (algebraic, arithmetic) solutions.

Question	Code	Definition
Quality	Correct (C)	The student solved the posed problem correctly. This category also includes work where the student applied the mathematical model correctly but made a calculation error.
	Incorrect (IC)	The learner misapplied the new material. The solution was coded IC even if the student gave one correct and one incorrect solution for their questions (e.g., the rule is misidentified)
	0	The student did not submit a solution.
Type of problem-solving	Alg	The students solve the problem using algebraic thinking (Kaput, 2008)
	Ar	The students solve the problem arithmetically (without symbols or equations)
	Alg+Ar	The student combines both types of thinking.

Table 4. Coding the quality and type of the solutions

The following example shows the procedure of coding students' work (see Figure 2, Figure 3, and Table 5) and the reasoning behind it according to the coding frame.

Example (S19): The coding of S19's work according to the coding frame:

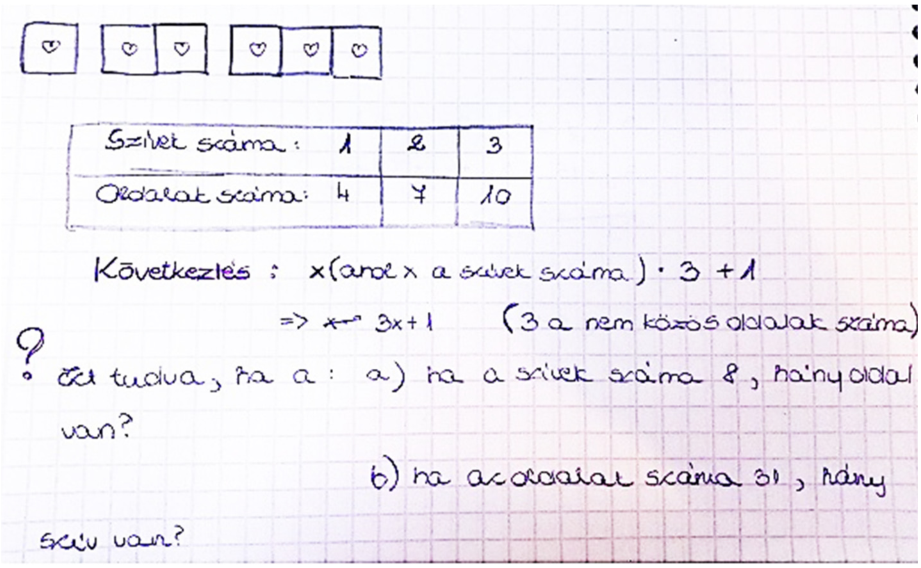


Figure 1. The task of PP by S 19

Translation:

Number of hearts	1	2	3
Number of segments	4	7	10

Table 5. Translation of the student's table shown in Figure 2

Conclusion: x (number of hearts [within one figure]) ·3 +1 => $3x + 1$, (3 is the number of [not coincident] segments)

Knowing this, a) if the number of hearts is 8, how many sides do we have? b) if the number of sides is 31, how many hearts do we have?

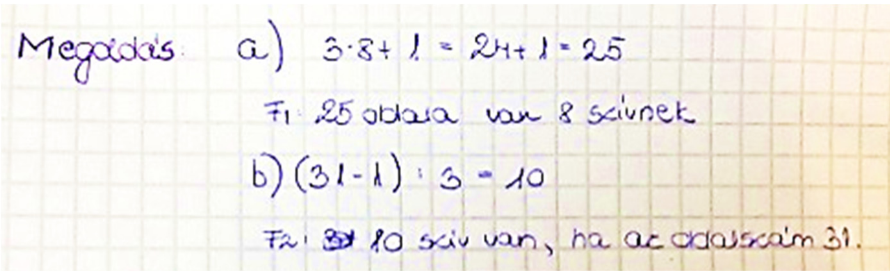


Figure 2. S19's solution

Solution:

a) F1: the 8th heart has 25 sides

b) F2:there are 10 hearts in case of 31 sides

Problem-posing quality: The posed problem is correct; two relevant questions were asked about the pattern.

Problem-posing question type: The first question is *F* (asking for a far element), and the second is *B* (the student requires the number of hearts, not sections).

Solution type and quality: *C* and *Alg+Ar*, i.e., the student solves the problem correctly, uses symbols to define the rule, and answers the first question by substituting 8 in the defined rule. In contrast, she answers the second question thinking backwards, not using an equation with the found rule.

RESULTS AND DISCUSSION

The focus of this paper is on analyzing the problem-solving conducted in the classroom discussions (*Task 1, 2, 3*) and students' problem-posing and problem-solving activity after the class discourse (*Task 3*). The first part of the classroom discourse was about finding the general rule, generalization ($n \rightarrow a_n$), while the second part presents students' relational way of thinking ($a_n \rightarrow n$).

Observing generalization through classroom discussions

The classroom discussion revealed the identified rules from the pair work phase. For example, in *Task 1*, students identified the following rules using such mathematical formulations:

> S 22: "The number of the segments increases by five."
>
> S 9: "The number of segments equals five times the number of the points plus one."
>
> S 39: "Let's regard the first one as 0, the second 1, and six times a number of the figure minus the before defined numbers."
>
> S 20: "Six times x minus (x – 1), where x stands for the number of points."
>
> S 15: "no. of segments = 5x + 1, where x represents the number of points".

All the rules were identified by the students. S20 was one of the students who included symbols in his solution, and then in the last case, the intervention of the teacher resulted in formulating the algebraic rule, which led to the possibility of forming equations as the Curriculum required. In this study, some students expressed their ideas verbally in a natural language in classroom discussions, facilitating generalization, whereas others were already confident using symbols. This observation aligns with the findings reported by Lócska and Kovács (2022).

Class discussion after the problem-posing part was different from the previous class discussions. The students became more confident in solving patterning tasks, revealing the general formula and using symbols, relying less and less on the help of the teacher. The following extract shows how the students solved one of their classmate's patterning problems.

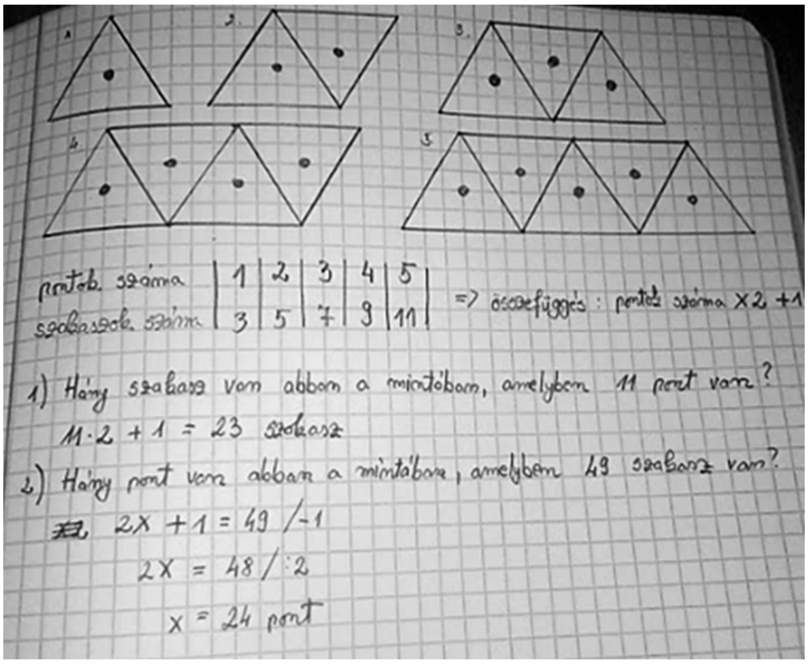


Figure 3. S32's problem

S37: If we multiply the number of segments by 3 and subtract 2, we get 5. […] the next is $3 \cdot 3 - 2 = 7$, then $3 \cdot 4 - 3 = 9$, but I don't know in general.

S56: I think is $3 \cdot x - 1$, as a general rule.

Teacher: Let's see if that is correct!

Many: [nods no]

S57: But if we subtract 1 from 9, we get 8. That is not true.

S33: $3 \cdot x - x$, we always have to subtract different numbers, don't we?

S45: But … $3 \cdot 2$ is 6, and we subtract 1, and then we subtract 2, and 3, and so on…

Teacher: So the number we subtract always…

S45: Increases by one!

S42: Yes. Can I say it? It is $2 \cdot x + 1$.

Teacher: Let's check it! […] It is okay, but what does it have to do with S37's solution? […]

S57: It is like $3 \cdot x - (x - 1)$, isn't it?

Teacher: And that is?

Many: $2x + 1$.

Observing relational thinking in classroom discussions

After solving *Task 1* individually, students presented and compared their solutions in the classroom discussion part. Based on their previous knowledge regarding equations and the recently found general rules we expected that they would introduce linear equations (*1.c.* 5x + 1 = 46 and *1.d.* 5x + 1 = 50). Analyzing the transcripts of the class discussion, we observed that students tended to solve these types of problems without equations (although this topic was taught at the 6th-grade level) using the method of working backwards (cover-up method in one step – see the following transcript). Based on Franke et al. (2008), we may call this solution method relational thinking. This way, students may treat actions on equations as sense-making activities.

S 32: We subtract one from 46, then divide it by 5, so 9 points.

T: Yes, how would you write it down mathematically?

S32: 46 minus 1 in brackets, divided by 5 [i.e., (46 – 1):5].

In this case, the equal sign expresses a relation, i.e., the student becomes aware of the relation between the two sides of the equation, without thinking of it as an actual equation. The same quantity stands on both sides of the equal sign, one symbolically (5x+1 – algebra), the other as a concrete number (46 – arithmetic). Therefore, this relational thinking of the students forms a bridge between arithmetics and algebra.

Observing students' problem-posing activities

Regarding problem posing as a window into students' mathematical understanding, some researchers claim that problem-posing has the potential to explore the nature of students' understanding of mathematics (Silver, 1994). We aimed to map students' way of thinking (algebraic or arithmetic) by analyzing their posed problems using the coding frame shown above (Tables 2, 3, 4). Table 6 shows the number of students for each quality type of question. Only six of the students who posed relevant questions gave an incorrect solution.

Code	No. of students
Relevant (R)	34
Irrelevant (Ir)	1
Neither (0)	7
No pattern (Np)	6
Total	48

Table 6. The quality of the posed questions

Analyzing the questions, we observed certain students' problem-posing and solving particularities: a) students who ask about only close elements b) students whose questions involve far elements c) how students solve backward problems.

22 students posed problems asking about one close element of the pattern and 3 of them asked about only close elements. We assume they were more comfortable with those simple calculations. This way they did not have to figure out the rule which fits all the numbers. The students whose first question was coded C (close

element), but the second with F/G (far/generalization) or B (backward thinking), were more likely to use an algebraic solution in their answers.

25 students received code F (far element) for one of their questions, 9 of them G (looking for a general rule). Table 7 shows the composition of the solutions given for questions coded G. It shows that students who pose questions that require a generalization tend to solve the problem algebraically, mostly with success.

Type \ Quality	Correct	Incorrect
Algebraic	7	1
Arithmetic	0	1

Table 7. Solution for the questions with G (generalization)

Analyzing the solutions for the questions coded with F (far element), we observed that solutions differ in this case: 8 students answered the question algebraically (all of them correctly), 8 arithmetically (3 out of them were incorrect), 2 students combined algebraic and arithmetic thinking correctly.

We were interested in analyzing backward problem solutions (coded B) and establishing what characterizes these solutions. In classroom discussions, we observed that these students preferred relational thinking. We were intrigued by the solutions that followed their posed problems. 18 students posed backward problems. Table 8 shows the distribution of the answers for questions coded B.

Type \ Quality	Correct	Incorrect
Algebraic (Alg)	10	1
Arithmetic (Ar)	4	1
Combined (Alg+Ar)	2	0

Table 8. Solution for the questions with B

13 out of 18 students tried to answer the question using algebraic formulas, symbols, or equations, although they preferred using none in the classroom discussion. We assume that the teacher's intervention regarding solutions that involved an equation had an impact on the students' answers. However, we might also think that in 7th grade, they are in transition from arithmetic to algebraic thinking, and relational thinking through pattering activities could serve as a bridge for them.

CONCLUSION

This study aimed to explore how algebraic thinking is reflected in students' problem-posing activities that involve patterning experiences. Analyzing the students' problem-solving activities in class discussions, we observed that they could formulate rules and discover connections. We deduced that although most students expressed their ideas verbally in everyday language this activity facilitated generalization. These findings correspond to Lócska and Kovács (2022).

We found that these participating 7th-grade students preferred relational thinking over equations before problem-posing activities, forming a bridge between arithmetic and algebra. Regarding students' problem-posing activities, we noticed that students who posed problems that require generalization tended to solve them algebraically, whereas students, who posed backward problems, used more algebraic solutions in their classroom discourse than before. We also observed that students became more confident in solving patterning-related tasks and connecting variables to them, as after the problem-posing activity their process of solving a classmate's problem hardly needed the teacher's help. It agrees with Rivera's (2013) work according to which more patterning activities deepen students' conceptions of the meaning of symbols, i.e. algebra, and also these conceptions undergo changes from the concrete to the abstract. Therefore we assume, that 7th-grade students are in a transition stage from arithmetic thinking to algebraic thinking, and as a result patterning activities provide a fruitful opportunity to observe generalizations before and during using variables, (see Schoenfeld and Arcavi, 1988). However, this study also has its limitations: the pandemic situation constrained the research in several respects. One of the factors was missing data, which was due to a reduced number of children participating in the problem-posing activity. Additionally, all 61 students, who took part in online lessons worked from home, sometimes in uncontrolled settings. Although these limitations stand, we believe we gained experience in mapping students' thinking.

In conclusion, we claim that we had the opportunity to map students' way of thinking through problem-solving and problem-posing. We managed to establish a window to observe different perspectives of students' thinking and also a possible way for teachers to foster algebraic thinking through patterning activities.

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Reports from the three workshops

PEDAGOGICAL ASPECTS OF PROBLEM SOLVING AND PROBLEM POSING IN CLASSROOM

Ioannis Papadopoulos & Nafsika Patsiala
Aristotle University of Thessaloniki, Greece

In this report a summary of the content of a workshop held at the 2022 ProMath conference is given. The focus of the workshop was on the pedagogical aspects of problem solving and problem posing in classroom. The whole discussion was developed around five axes: (i) differentiated vs direct instruction, (ii) individual vs group work, (iii) problem solving and problem posing as a goal vs a tool for teaching, (iv) problem solving after problem posing, and (iv) how the previous four choices vary across educational levels.

SUMMARY OF THE WORKSHOP

The first thing considered necessary was to agree on a common understanding of what the term 'pedagogical aspects' might mean. This first round of discussion resulted in considering these aspects as relevant to the numerous decisions the teacher should make while preparing for a problem-solving or problem-posing based teaching session in classroom. The next step was to identify specific topics that are relevant to these pedagogical aspects. At the end the discussion was limited to five of them. Should the teacher follow a direct or a differentiated problem-solving and/or problem-posing instruction in classroom? Is it the individual work or the collaborative one that really brings benefits to students? Should problem solving and problem posing being the aim of teaching (in the sense of acquiring certain skills on solving and posing problems) or is it better for them to be used as vehicles to teach new concepts? Does it make sense to solve the problems produced during a problem-posing session? Finally, given that the very same questions are met across all educational level could there be the same answer on these questions for all the educational levels?

1. Is it better to use the same problem/situation for all the students or follow a differentiated approach?

It was not easy for the participants to tackle the range of aspects of this issue given the differences between problem solving and problem posing. They agreed that there must be a different answer for each kind of activities. So, for problem solving it is considered appropriate to use the differentiated approach giving the students problems according to their skills and needs. The problems addressed to the students can be either modified versions of the same problem but with different level of complexity or a collection of entirely different type of problems.

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On the other hand, for a problem-posing session, it is preferable to use a common initial problem or situation, since in this case students create their own problems, expressing thus themselves and their mathematical ideas in various ways. Therefore, it is not an issue that the same problem is used since the outcome is different for each student.

2. Is individual or group work more beneficial?

Another important pedagogical aspect that needs to be taken under consideration is whether the students should work individually on collaboratively. There are advantages and disadvantages in both individual and group work. For example, teachers must consider the fact that understanding of mathematical concepts at the individual and/or group level might vary significantly, and this probably has an impact in the problem-solving or problem-posing process. On the one hand, when students work individually, they have the opportunity for personal growth, and they are given space and time to approach the situation in any way they consider fit better. On the other hand, working in groups helps the students exchange ideas, which will be discussed among the group members enhancing thus their social metacognitive skills. The participants gave emphasis to the risk for some students to avoid participating in collaborative work due to the uncertainty they feel and the insecurity to express and negotiate their ideas.

3. Should problem solving and problem posing be used as a means or a goal of instruction?

We need to see problem solving and problem posing as both a means of instruction and a goal of instruction. As a goal of teaching, they could help students to get acquainted with the use of strategies and applying them consciously facilitating thus the development of the competence to solve and pose problems and the development of the habit of seeking and using structure. As a means of teaching, problem solving and problem posing could provide an alternative approach to introduce students to new concepts of mathematics resulting in a deep understanding of mathematical content.

4. Is it useful to solve the problems after posing them or not?

It is without a doubt that problem solving is of great importance. But when it comes to problem posing, is it useful for students to solve the posed problems? On the one hand, knowing that they have to solve afterwards the problems, they may feel stressed about it and decide to pose problems that are familiar to them and easy to solve. Therefore, it is possible to produce textbook like problems deprived of originality, creativity and diverse thinking. On the other hand, if the posed problem won't be considered in terms of their solvability a danger is lurking for the students to ignore the structure of the given problem as the starting point for forming new ones. This might result in problems that do not make sense from the mathematical point of view. The participants suggested a balance between both, according to the specific circumstances.

5. Do these decisions differ across educational levels?

The starting point for the discussion was the acknowledgment that it is necessary to consider all these aspects when problem-solving and problem-posing sessions take place in classroom, no matter the educational level (primary, secondary, tertiary level). The dilemmas may be the same but quite often the answers differ according to the circumstances. For example, as far as the second dilemma is concerned, the large audiences in university courses do not allow the use of small groups or differentiated teaching. But, in relation to the third dilemma, problem solving and problem posing can be used as both a means and a goal across all educational levels.

REPORT OF THE WORKSHOP ON THE SKILLS RELATED TO PROBLEM SOLVING AND POSING

Eszter Kónya

University of Debrecen, MTA-Rényi-ELTE Research Group in Mathematics Education, Hungary

The workshop aimed to discuss skills that can be developed through an interplay between problem solving and problem posing.

The work involves seven researchers from four different countries:

Emine Gül Celebi-Ilhan from Ankara, Turkey; Marianthi Zioga from Thessaloniki, Greece; Jasmina Milinkovic from Belgrade, Serbia; András Ambrus from Budapest, Hungary and Emőke Báró, Gabriella Babcsányi-Tóth, Eszter Kónya from Debrecen, Hungary.

As a first step, an attempt was made to situate the two key activities, problem solving (PS) and problem posing (PP), in the classroom teaching-learning process. For this purpose, we have created a schematic diagram (Figure 1) where PS and PP, among other activities, contribute to the main objective of the mathematics lessons, which is to teach basic knowledge.

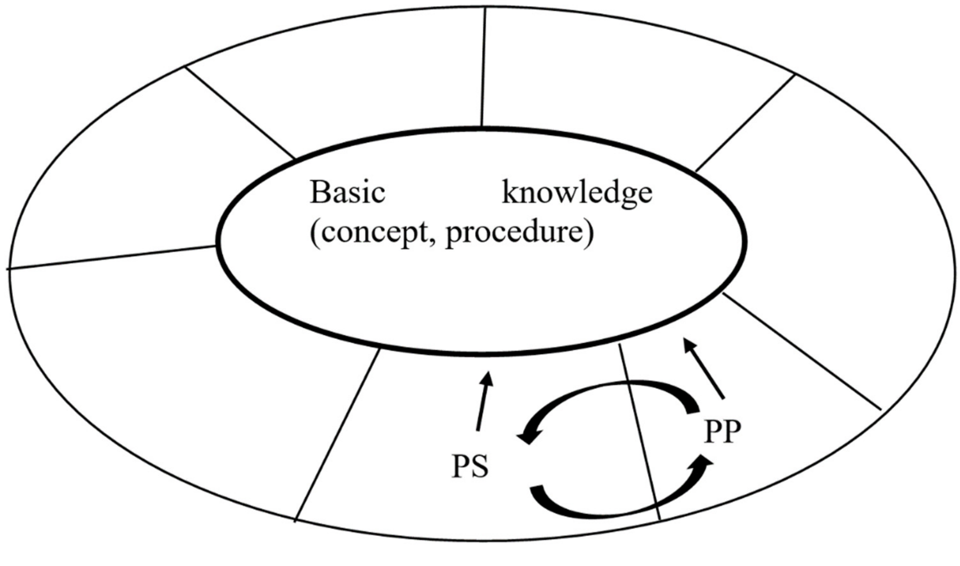

Figure 1. The place of PS and PP activities in the classroom

We agreed that problem solving and posing activities should always be combined and considered a unity. Starting from problem solving, we can naturally move on to problem posing. We can modify, specialize, or generalize the original problem during or after PS. Conversely, if the task is to pose a new problem, the students

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should also solve it. Of course, if the pupils come up with too many problems, the teacher must choose the one that is worth solving.

We have differentiated between PS and PP in terms of whether it takes the form of individual or group work. The focus is on developing individual cognitive skills in the first case, while social skills are also produced in the second case (Figure 2).

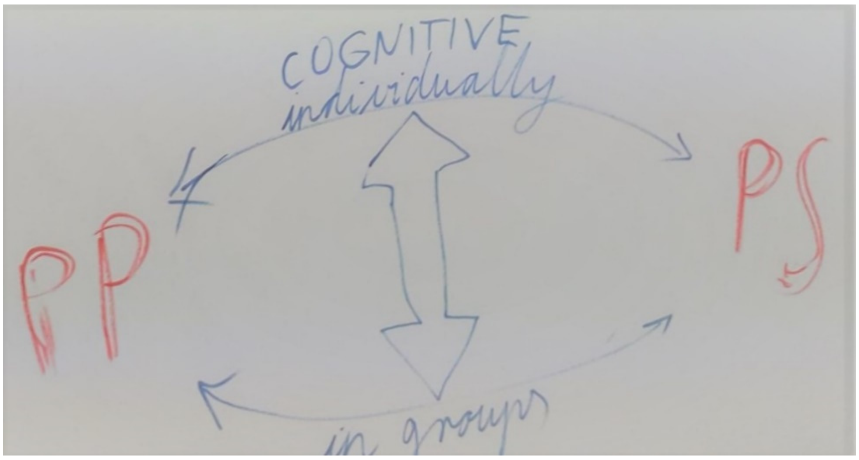

Figure 2. There is a difference between working individually or in groups.

We reviewed the skills that can be developed through PS and PP activities by the associated cognitive and behavioral manifestations. Mental manifestations are modeling, using heuristics, reasoning, critical thinking, control, and reflection. Communication, brainstorming, discussion, and assessment are rather part of social activities. We thought a lot about which skills we could link directly to the above and finally settled on the following. We identified creativity, flexibility, adaptability, and the ability to recognize the transfer between different mathematical topics as cognitive skills. In addition, the ability to argue, the ability to think critically, and the ability to express one's ideas accurately were highlighted as communication skills.

The final question in the workshop was whether it is possible to establish some order in developing the skills listed. However, we left this question open and only stated that skills that can develop in each lesson depend mainly on the problem's mathematical content and the classroom climate.

PROBLEM SOLVING AND PROBLEM POSING: THE QUESTIONS OF LOCATION AND TIME ALLOCATION IN SCHOOL MATHEMATICS

Inga Gebel and Ana Kuzle[1]

University of Potsdam, Germany

In this report, a summary on the contents of the workshop is given. In particular, two questions were dealt with, namely whether problem solving and problem posing should be autonomous subjects or should be integrated into daily mathematics teaching, as well as the issue of time allocation. To structure our discussion, we gathered advantages and disadvantages for both scenarios which we present below. Finally, we provide an overall assessment of the discussion.

SUMMARY OF THE WORKSHOP

Focus of the workshop and its participants

The workshop was guided by the following two questions: Problem solving and problem posing:

1. Autonomous subjects in classroom or integrated in daily mathematics teaching?
2. What about the issue of time allocation?

The participants of the workshop group were: Branka Antunović-Piton (Juraj Dobrila University of Pula, Croatia), Christos Souralis (National and Kapodistrian University of Athens, Greece), Georgios Thoma (Loughborough University, United Kingdom), and Inga Gebel and Ana Kuzle (University of Potsdam, Germany). Given that the participants came from four different educational systems, this provided various outtakes on the two workshop questions.

In the following, problem-solving lessons are understood as lessons during which both problem solving and problem posing activities can be integrated.

Problem solving and problem posing as autonomous subjects

The notion of problem solving as a *goal* of teaching is well-known. Here, problem solving is dealt as an additional topic, somewhat an addendum to school contents that need to be taught (see Rott & Papadopoulos, 2019). Having problem-solving

1. The authors are listed in alphabetical order and contributed equally to this report.

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lessons being regularly and systematically included in school (mathematics) lessons, could bring numerous advantages.

Firstly, this would allow allocating extra time for each student for various problem-solving experiences. Already from the literature (e.g., Donaldson, 2011) we know that for this complex cognitive process many practices are important for helping students grow in their problem-solving ability, such as osmosis (give lots of problems), give "good" problems, memorization (teach specific or general heuristic strategies (heurisms)), imitation (model problem solving), cooperation (limit teacher input by having students work in small groups), reflection (promote metacognition by asking metacognitive questions or encouraging students to be reflective), and highlight multiple solutions. This approach, however, may not be appealing to teachers since the lack of time is often reported as one of the biggest obstacles regarding the implementation of problem solving (Kuzle et al., 2023).

Secondly, the position of problem solving for learning in general would become clearer through visibility in curricula, and, with it, in teaching plans and would not be neglected. Thirdly, in terms of interdisciplinary learning, problem solving would not necessarily be limited exclusively to mathematical problems. Different heurisms could be taught in the context of projects, which could be applied in different disciplines (for example, systematic working).

However, also these advantages would be met with some difficulties. Although additional time would be given to problem solving, teachers need to be prepared for this, and class time must be available, which is contrary to the framework of many countries due to teacher shortages. Even though interdisciplinary teaching of heurisms can lead to more networked knowledge, this does not automatically ensure that mathematics-specific heurisms can be adequately addressed and connected to current learning. The problem here is the concern that problem-solving skills would be taught in a too general, and random manner.

Problem solving and problem posing integrated in daily mathematics teaching

Problem solving as a teaching *method* (i.e., teaching content-related topics by using problem solving) is also well-known and already discussed for decades (e.g., see Rott & Papadopoulos, 2019). Such approach could also bring numerous advantages, and has many proponents. Amongst others, Winter (1995) sees problem solving as one of the three basic experiences of mathematics teaching, so that the disciplinary reference should be clearly emphasized.

Some heurisms have only special application in mathematics, and, hence, can only be learnt in the context of mathematics problem solving (e.g., invariance principle, symmetry principle). This requires that pre-service teachers' educational programs prepare mathematics teachers in this regard, and also learn to use the curricula as a basis for planning their mathematics lessons. Therefore, it is desirable that problem solving be explicitly listed in the mathematics curricula, which is the case in many countries, but in a manner that would reflect problem solving as an integral part of mathematics or as a habit of mind. Moreover, the curricula worldwide need to reflect a network between both content and process competencies.

Mathematical content can be learned through problem solving (Donaldson, 2011). Problem solving should therefore not be separated from content but rather connected to it. Additionally, through problem solving, students can have an incentive to engage with mathematics. However, "the availability of problems suited to convey mathematical contents as well as the cognitive load of dealing with problems in addition to learning new contents" (Rott & Papadopoulos, 2019, p. 216) is problematic for both teachers as well as students.

Even though, problem solving is currently integrated into mathematics curricula and mathematics instruction, studies (e.g., Kuzle et al., 2023) show that implementation should still be significantly optimized. This means that the potential of this competence is not yet fully exploited in school mathematics. Obstacles are seen, among other things, in the fact that problem solving is perceived by teachers as an additional task alongside the teaching of mathematical content. Thus, teachers' perspectives on different notions of problem solving needs to be challenged, and expanded.

Change in the teaching culture

In the workshop, we agreed that rather than a change in the location of problem-solving lessons and its time allocation, what is needed is a general change in the teaching culture itself. Instead of teaching content according to "teaching to the test", it is desirable that problem solving is seen as a basic experience of mathematics education. At the same time, fundamental questions regarding assessment in mathematics education arose; should problem-solving competence be tested in the same way as mathematical content or should performance assessment in mathematics generally be more process-oriented?

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